On Axiomatic Approaches to Intertwining Operator Algebras

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Abstract

We study intertwining operator algebras introduced and constructed by Huang. In the case that the intertwining operator algebras involve intertwining operators among irreducible modules for their vertex operator subalgebras, a number of results on intertwining operator algebras were given in [H9] but some of the proofs were postponed to an unpublished monograph. In this paper, we give the proofs of these results in [H9] and we formulate and prove results for general intertwining operator algebras without assuming that the modules involved are irreducible. In particular, we construct fusing and braiding isomorphisms for general intertwining operator algebras and prove that they satisfy the genus-zero Moore-Seiberg equations. We show that the Jacobi identity for intertwining operator algebras is equivalent to generalized rationality, commutativity and associativity properties of intertwining operator algebras. We introduce the locality for intertwining operator algebras and show that the Jacobi identity is equivalent to the locality, assuming that other axioms hold. Moreover, we establish that any two of the three properties, associativity, commutativity and skew-symmetry, imply the other (except that when deriving skew-symmetry from associativity and commutativity, more conditions are needed). Finally, we show that three definitions of intertwining operator algebras are equivalent.

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1 Introduction

The theory of vertex operator algebras and their representations provides the natural foundation and context for a wide range of structures and concepts in mathematics and physics, such as the Fischer-Griess Monster sporadic finite simple group and monstrous moonshine, representation theory of affine Kac-Moody algebras and the Virasoro algebra, knot and three-dimensional manifold invariants, conformal and topological field theories, and topological quantum computation. The notion of vertex (operator) algebra was introduced in mathematics by Borcherds [B] and Frenkel, Lepowsky and Meurman [FLM]. In the physics

literature, Belavin, Polyakov and Zamolodchikov [BPZ] formalized and studied the operator product algebra structure in conformal field theory and the notion of chiral algebra in physics (see e.g. [MS]) essentially coincides with the notion of vertex operator algebra.

In the study of representation theory of vertex operator algebras and conformal field theory, intertwining operators (or chiral vertex operators in physics) are one of the main interesting objects. The important notions of fusion rule, fusing matrix, braiding matrix, and Verlinde algebra for a vertex operator algebra or a conformal field theory are all defined in terms of intertwining operators (see [V], [TK], [MS], [FHL], [H9]). Intertwining operators also give field-theoretic description of nonabelian anyons. The direct sum of all inequivalent irreducible modules for a suitable vertex operator algebra, equipped with intertwining operators, has a natural algebraic structure called intertwining operator algebra (see [H3], [H6] and [H9]), which is a natural generalization of the definition of vertex operator algebras. In the special case that the fusion rules are structure constants of group algebras of abelian groups, a notion of abelian intertwining operator algebra was introduced in [DL1, DL2] and examples were constructed in [DL2, DL3, FRW]. In general, several definitions of intertwining operator algebras were given in [H].

The intertwining operator algebras are multivalued generalizations of vertex operator algebras. They were first defined using the convergence property, associativity and skew-symmetry as the main axioms. These algebras are equivalent to genus-zero chiral rational conformal field theories. The representation theory of vertex operator algebras, especially the techniques developed in the tensor category theory (see [HL1]– [HL7], [H3]), provides an effective way to construct these algebras (see [H4], [H6], [H8], [HL9] for details).

In the present paper, we study intertwining operator algebras introduced and studied in [H3], [H6] and [H9]. In the case that the intertwining operator algebras involve intertwining operators among irreducible modules for their vertex operator subalgebras, intertwining operator algebras were studied in [H9] but some of the proofs were postponed to an unpublished monograph [H10]. In this paper, we give the proofs of these results in [H9] and we formulate and prove results for general intertwining operator algebras without assuming that the modules involved are irreducible. In particular, we construct fusing and braiding isomorphisms for general intertwining operator algebras and prove that they satisfy the genus-zero Moore-Seiberg equations. We show that the Jacobi identity for intertwining operator algebras is equivalent to generalized rationality, commutativity and associativity properties of intertwining operator algebras. We introduce the locality for intertwining operator algebras and show that the Jacobi identity is equivalent to the locality, assuming that other axioms hold. Moreover, we establish that any two of the three properties, associativity, commutativity and skew-symmetry, imply the other (except that when deriving skew-symmetry from associativity and commutativity, more conditions are needed). Finally, we show that three definitions of intertwining operator algebras are equivalent.

This paper is organized as follows. In Section 2, we recall the definition of intertwining

operator algebras and some results obtained in [H9]. We give a detailed description of the fusing isomorphism and braiding isomorphism obtained from the associativity and the skew-symmetry. In Section 3, we derive the relations among the Jacobi identity, the duality properties and the locality. In Section 4, we derive isomorphisms between quotient vector spaces obtained from the tensor products of the vector spaces consisting of the same type of intertwining operators. Moreover, we prove that these isomorphisms satisfy the genus-zero Moore-Seiberg equations. In Section 5, we prove the equivalence of the definitions given in [H9].

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2 Review of the definitions and properties

In this section, we review the definitions and basic properties in the theory of intertwining operator algebras in [H9]. We also give a detailed description of the fusing isomorphism and braiding isomorphism for general intertwining operator algebras.

2.1 Formal calculus and complex analysis

We first recall some basic notations and facts in formal calculus and complex analysis. See [FLM, FHL, H9] for more details.

In this paper, x, $x_0, \dots$ are independent commuting formal variables, and for a vector space W and a formal variable x, we shall denote

$$
\begin{aligned}
W[x] &= \left\{ \sum_{n\in\mathbb{N}} w_n x^n \mid w_n \in W,\ \text{all but finitely many } w_n = 0 \right\},\\
W[x, x^{-1}] &= \left\{ \sum_{n\in\mathbb{Z}} w_n x^n \mid w_n \in W,\ \text{all but finitely many } w_n = 0 \right\},\\
W[[x]] &= \left\{ \sum_{n\in\mathbb{N}} w_n x^n \mid w_n \in W \right\},\\
W[[x, x^{-1}]] &= \left\{ \sum_{n\in\mathbb{Z}} w_n x^n \mid w_n \in W \right\},\\
W((x)) &= \left\{ \sum_{n\in\mathbb{Z}} w_n x^n \mid w_n \in W, w_n = 0 \text{ for sufficiently negative } n \right\},
\end{aligned}
$$

$$W\{x\} = \left\{ \sum_{n\in\mathbb{C}} w_n x^n \mid w_n \in W \right\},$$

and we shall also use similar notations for series with more than one formal variables. For any $f(x) \in W\{x\}$, we shall use $\mathrm{Res}_x f(x)$ to denote the coefficient of x^{-1} in $f(x)$. We shall use $z, z_0, \ldots,$ to denote complex numbers, *not* formal variables.

Let

$$\delta(x) = \sum_{n\in\mathbb{Z}} x^n. \tag{2.1}$$

This "formal δ-function" has the following simple and fundamental property: For any $f(x) \in \mathbb{C}[x, x^{-1}]$,

$$f(x)\delta(x) = f(1)\delta(x). \tag{2.2}$$

This property has many important variants. For example, for any

$$X(x_1, x_2) \in (\mathrm{End}\, W)[[x_1, x_1^{-1}, x_2, x_2^{-1}]] \tag{2.3}$$

(where W is a vector space) such that

$$\lim_{x_1\to x_2} X(x_1, x_2) = X(x_1, x_2)\Big|_{x_1=x_2} = X(x_2, x_2) \tag{2.4}$$

exists, we have

$$X(x_1, x_2)\delta\left(\frac{x_1}{x_2}\right) = X(x_2, x_2)\delta\left(\frac{x_1}{x_2}\right). \tag{2.5}$$

The existence of the "algebraic limit" defined in (2.4) means that for an arbitrary vector $w \in W$, the coefficient of each power of x_2 in the formal expansion $X(x_1, x_2)w|_{x_1=x_2}$ is a finite sum. We use the convention that negative powers of a binomial are to be expanded in nonnegative powers of the second summand. For example,

$$x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) = \sum_{n\in\mathbb{Z}} \frac{(x_1 - x_2)^n}{x_0^{n+1}} = \sum_{m\in\mathbb{N},\, n\in\mathbb{Z}} (-1)^m \binom{n}{m} x_0^{-n-1} x_1^{n-m} x_2^m. \tag{2.6}$$

We have the following identities:

$$x_1^{-1}\delta\left(\frac{x_2 + x_0}{x_1}\right) = x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right), \tag{2.7}$$

$$x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) = x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right). \tag{2.8}$$

Let $\mathbb{C}[x_1, x_2]_S$ be the ring of rational functions obtained by inverting the products of (zero or more) elements of the set S of nonzero homogenous linear polynomials in x_1 and x_2.

Also, let ι_{12} be the operation of expanding an element of $\mathbb{C}[x_1, x_2]_S$, that is, a polynomial in x_1 and x_2 divided by a product of homogenous linear polynomials in x_1 and x_2, as a formal series containing at most finitely many negative powers of x_2 (using binomial expansions for negative powers of linear polynomials involving both x_1 and x_2); similarly for ι_{21}, and so on. The following fact from [FHL] will be very useful:

Proposition 2.1. *Consider a rational function of the form*

$$f(x_0, x_1, x_2) = \frac{g(x_0, x_1, x_2)}{x_0^r x_1^s x_2^t}, \tag{2.9}$$

where g is a polynomial and $r, s, t \in \mathbb{Z}$. Then

$$x_1^{-1}\delta\left(\frac{x_2 + x_0}{x_1}\right)\iota_{20}(f|_{x_1=x_0+x_2}) = x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right)\iota_{10}(f|_{x_2=x_1-x_0}) \tag{2.10}$$

and

$$\begin{aligned} x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right)\iota_{12}(f|_{x_0=x_1-x_2}) - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right)\iota_{21}(f|_{x_0=x_1-x_2}) \\ = x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right)\iota_{10}(f|_{x_2=x_1-x_0}). \end{aligned} \tag{2.11}$$

For any $\mathbb{Z}$-graded, or more generally, $\mathbb{C}$-graded, vector space $W = \coprod_n W_{(n)}$, we use

$$W' = \coprod_n W_{(n)}^* \tag{2.12}$$

to denote its graded dual.

For any $z \in \mathbb{C}$, we shall always choose $\log z$ so that

$$\log z = \log|z| + i\arg z \ \text{ with } \ 0 \le \arg z < 2\pi. \tag{2.13}$$

Given two multivalued functions f_1 and f_2 on a region, we say that f_1 *and* f_2 *are equal* if on each simply connected open subset of the region, for any single-valued branch of f_1, there exists a single-valued branch of f_2 equal to it, and vice versa.

2.2 Intertwining operator algebras and some consequences

In this part, we recall basic notions and results in the theory of intertwining operator algebras. We assume that the reader is familiar with the basic definitions and properties of vertex operator algebras, their modules and intertwining operators. For the details of these definitions and properties, the reader is referred to [FHL,FLM,H9]. See also [HL8,H1,H2,H7] for the equivalences of different approaches to vertex operator algebras.

Let $(V, Y, \mathbf{1}, \omega)$ be a vertex operator algebra, and let W_1, W_2, W_3 be modules of V. We denote the space of the intertwining operators of the type $\binom{W_3}{W_1\ W_2}$ by $\bar{\mathcal{V}}_{W_1W_2}^{W_3}$ instead of $\mathcal{V}_{W_1W_2}^{W_3}$, which we shall use to denote a subspace later in the definition of intertwining operator algebra. The dimension of this vector space is the fusion rule of the same type and is denoted by $\bar{\mathcal{N}}_{W_1W_2}^{W_3}$.

Let $\mathcal{Y}$ be an intertwining operator of type $\binom{W_3}{W_1W_2}$. For any $w_{(1)} \in W_1$, we shall use $\mathcal{Y}_{(n)}(w_{(1)})$ to denote $\mathrm{Res}_x x^n \mathcal{Y}(w_{(1)}, x),\ n \in \mathbb{C}$, that is,

$$\mathcal{Y}(w_{(1)}, x) = \sum_{n\in\mathbb{C}} \mathcal{Y}_{(n)}(w_{(1)})x^{-n-1}. \tag{2.14}$$

The $L(-1)$ derivative property

$$\frac{d}{dx}\mathcal{Y}(w_{(1)}, x) = \mathcal{Y}(L(-1)w_{(1)}, x) \tag{2.15}$$

and the $L(-1)$-conjugation property

$$[L(-1), \mathcal{Y}(w_{(1)}, x)] = \mathcal{Y}(L(-1)w_{(1)}, x) \tag{2.16}$$

of intertwining operators for $w_{(1)} \in W_1$ will be used frequently, where the operator $L(-1)$ acts on three different modules.

For any complex number ζ and any $w_{(1)} \in W_1$, $\mathcal{Y}(w_{(1)}, y)\Big|_{y^n=e^{n\zeta}x^n,\ n\in\mathbb{C}}$ is also a well-defined element of $\mathrm{Hom}(W_2, W_3)\{x\}$. We denote this element by $\mathcal{Y}(w_{(1)}, e^{\zeta}x)$. Note that this element depends on ζ, not just on e^{ζ}. Given any $r \in \mathbb{Z}$, we define

$$\Omega_r(\mathcal{Y}) : W_2 \otimes W_1 \to W_3\{x\} \tag{2.17}$$

by the formula

$$\Omega_r(\mathcal{Y})(w_{(2)}, x)w_{(1)} = e^{xL(-1)}\mathcal{Y}(w_{(1)}, e^{(2r+1)\pi i}x)w_{(2)} \tag{2.18}$$

for $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$. The following result was proved in [HL5]:

Proposition 2.2. *The operator $\Omega_r(\mathcal{Y})$ is an intertwining operator of type $\binom{W_3}{W_2\ W_1}$. Moreover,*

$$\Omega_{-r-1}(\Omega_r(\mathcal{Y})) = \Omega_r(\Omega_{-r-1}(\mathcal{Y})) = \mathcal{Y}. \tag{2.19}$$

In particular, the correspondence $\mathcal{Y} \mapsto \Omega_r(\mathcal{Y})$ defines a linear isomorphism from $\bar{\mathcal{V}}_{W_1W_2}^{W_3}$ to $\bar{\mathcal{V}}_{W_2W_1}^{W_3}$, and we have

$$\bar{\mathcal{N}}_{W_1W_2}^{W_3} = \bar{\mathcal{N}}_{W_2W_1}^{W_3}. \tag{2.20}$$

The first definition of intertwining operator algebras in [H9] is:

Definition 2.3 (Intertwining operator algebra). An *intertwining operator algebra of central charge* $c \in \mathbb{C}$ consists of the following data:

1. A vector space
$$W = \coprod_{a\in\mathcal{A}} W^a \tag{2.21}$$
graded by a finite set $\mathcal{A}$ containing a special element e (graded by *color*).

2. A vertex operator algebra structure of central charge c on W^e, and a W^e-module structure on W^a for each $a \in \mathcal{A}$.

3. A subspace $\mathcal{V}_{a_1a_2}^{a_3}$ of the space of all intertwining operators of type $\binom{W^{a_3}}{W^{a_1}W^{a_2}}$ for each triple $a_1, a_2, a_3 \in \mathcal{A}$, with its dimension denoted by $\mathcal{N}_{a_1a_2}^{a_3}$.

These data satisfy the following axioms for any $a_1, a_2, a_3, a_4, a_5, a_6 \in \mathcal{A}$, $w_{(a_i)} \in W^{a_i}$, $i = 1, 2, 3$, and $w'_{(a_4)} \in (W^{a_4})'$:

1. The W^e-module structure on W^e is the adjoint module structure. For any $a \in \mathcal{A}$, the space $\mathcal{V}_{ea}^{a}$ is the one-dimensional vector space spanned by the vertex operator for the W^e-module W^a. For any $a_1, a_2 \in \mathcal{A}$ such that $a_1 \neq a_2$, $\mathcal{V}_{ea_1}^{a_2} = 0$.

2. *Weight condition*: For any $a \in \mathcal{A}$ and the corresponding module $W^a = \coprod_{n\in\mathbb{C}} W^a_{(n)}$ graded by the action of $L(0)$, there exists $h_a \in \mathbb{R}$ such that $W^a_{(n)} = 0$ for $n \notin h_a + \mathbb{Z}$.

3. *Convergence properties*: For any $m \in \mathbb{Z}_+$, $a_i, b_j \in \mathcal{A}$, $w_{(a_i)} \in W^{a_i}$, $\mathcal{Y}_i \in \mathcal{V}_{a_i\, b_{i+1}}^{b_i}$, $i = 1, \ldots, m$, $j = 1, \ldots, m+1$, $w'_{(b_1)} \in (W^{b_1})'$ and $w_{(b_{m+1})} \in W^{b_{m+1}}$, the series
$$\langle w'_{(b_1)}, \mathcal{Y}_1(w_{(a_1)}, x_1) \cdots \mathcal{Y}_m(w_{(a_m)}, x_m) w_{(b_{m+1})}\rangle_{W^{b_1}}\big|_{x_i^n = e^{n\log z_i},\ i=1,\ldots,m,\ n\in\mathbb{R}} \tag{2.22}$$
is absolutely convergent when $|z_1| > \cdots > |z_m| > 0$ and its sum can be analytically extended to a multivalued analytic function on the region given by $z_i \neq 0$, $i = 1, \ldots, m$, $z_i \neq z_j$, $i \neq j$, such that for any set of possible singular points with either $z_i = 0$, $z_i = \infty$ or $z_i = z_j$ for $i \neq j$, this multivalued analytic function can be expanded near the singularity as a series having the same form as the expansion near the singular points of a solution of a system of differential equations with regular singular points. For any $\mathcal{Y}_1 \in \mathcal{V}_{a_1a_2}^{a_5}$ and $\mathcal{Y}_2 \in \mathcal{V}_{a_5a_3}^{a_4}$, the series
$$\langle w'_{(a_4)}, \mathcal{Y}_2(\mathcal{Y}_1(w_{(a_1)}, x_0) w_{(a_2)}, x_2) w_{(a_3)}\rangle_{W^{a_4}}\big|_{x_0^n = e^{n\log(z_1-z_2)},\ x_2^n = e^{n\log z_2},\ n\in\mathbb{R}} \tag{2.23}$$
is absolutely convergent when $|z_2| > |z_1 - z_2| > 0$.

4. *Associativity*: For any $\mathcal{Y}_1 \in \mathcal{V}_{a_1a_5}^{a_4}$ and $\mathcal{Y}_2 \in \mathcal{V}_{a_2a_3}^{a_5}$, there exist $\mathcal{Y}_{3,i}^a \in \mathcal{V}_{a_1a_2}^a$ and $\mathcal{Y}_{4,i}^a \in \mathcal{V}_{aa_3}^{a_4}$ for $i = 1, \ldots, \mathcal{N}_{a_1a_2}^a \mathcal{N}_{aa_3}^{a_4}$ and $a \in \mathcal{A}$, such that the (multivalued) analytic function

$$\langle w'_{(a_4)}, \mathcal{Y}_1(w_{(a_1)}, x_1)\mathcal{Y}_2(w_{(a_2)}, x_2)w_{(a_3)}\rangle_{W^{a_4}}|_{x_1=z_1, x_2=z_2} \tag{2.24}$$

defined on the region $|z_1| > |z_2| > 0$ and the (multivalued) analytic function

$$\sum_{a\in\mathcal{A}} \sum_{i=1}^{\mathcal{N}_{a_1a_2}^a \mathcal{N}_{aa_3}^{a_4}} \langle w'_{(a_4)}, \mathcal{Y}_{4,i}^a(\mathcal{Y}_{3,i}^a(w_{(a_1)}, x_0)w_{(a_2)}, x_2)w_{(a_3)}\rangle_{W^{a_4}}\bigg|_{x_0=z_1-z_2, x_2=z_2} \tag{2.25}$$

defined on the region $|z_2| > |z_1 - z_2| > 0$ are equal on the intersection $|z_1| > |z_2| > |z_1 - z_2| > 0$.

5. *Skew-symmetry*: The restriction of Ω_{-1} to $\mathcal{V}_{a_1a_2}^{a_3}$ is an isomorphism from $\mathcal{V}_{a_1a_2}^{a_3}$ to $\mathcal{V}_{a_2a_1}^{a_3}$.

Remark 2.4. To make our study slightly easier, we require in the present paper that the intertwining operator algebras satisfy the second axiom. This axiom is in fact a very minor restriction and in addition, it can be deleted from the definition and all the results of the intertwining operator algebras shall still hold.

Remark 2.5. *The skew-symmetry isomorphism $\Omega_{-1}(a_1, a_2; a_3)$ for all $a_1, a_2, a_3 \in \mathcal{A}$ give an isomorphism*

$$\Omega_{-1} : \coprod_{a_1,a_2,a_3\in\mathcal{A}} \mathcal{V}_{a_1a_2}^{a_3} \to \coprod_{a_1,a_2,a_3\in\mathcal{A}} \mathcal{V}_{a_1a_2}^{a_3} \tag{2.26}$$

and we still call this isomorphism the skew-symmetry isomorphism. In this paper, for simplicity, we shall omit subscript -1 in $\Omega_{-1}(a_1, a_2; a_3)$, $a_1, a_2, a_3 \in \mathcal{A}$, and in Ω_{-1} and denote them simply by $\Omega(a_1, a_2; a_3)$ and Ω, respectively.

The intertwining operator algebra just defined is denoted by

$$(W, \mathcal{A}, \{\mathcal{V}_{a_1a_2}^{a_3}\}, \mathbf{1}, \omega) \tag{2.27}$$

or simply W.

For the intertwining operator algebra, we have a second associativity and commutativity, which were proved in [H5]. To make this paper more complete, we shall rewrite the proof here.

Proposition 2.6 (Second Associativity). *Let $(W, \mathcal{A}, \{\mathcal{V}_{a_1a_2}^{a_3}\}, \mathbf{1}, \omega)$ be an intertwining operator algebra. Then we have the following second associativity: For any $a_1, a_2, a_3, a_4, a_5 \in \mathcal{A}$, $\mathcal{Y}_1 \in \mathcal{V}_{a_1a_2}^{a_5}$ and $\mathcal{Y}_2 \in \mathcal{V}_{a_5a_3}^{a_4}$, there exist $\mathcal{Y}_{3,j}^a \in \mathcal{V}_{a_1a}^{a_4}$ and $\mathcal{Y}_{4,j}^a \in \mathcal{V}_{a_2a_3}^a$ for $j = 1, \ldots, \mathcal{N}_{a_1a}^{a_4}\mathcal{N}_{a_2a_3}^a$*

and $a \in \mathcal{A}$, such that for $w_{(a_i)} \in W^{a_i}$, $i = 1, 2, 3$, and $w'_{(a_4)} \in (W^{a_4})'$, the (multivalued) analytic function

$$\langle w'_{(a_4)}, \mathcal{Y}_2(\mathcal{Y}_1(w_{(a_1)}, x_0)w_{(a_2)}, x_2)w_{(a_3)}\rangle_{W^{a_4}}|_{x_0=z_1-z_2, x_2=z_2} \tag{2.28}$$

defined on the region $|z_2| > |z_1 - z_2| > 0$ and the (multivalued) analytic function

$$\sum_{a\in\mathcal{A}} \sum_{j=1}^{\mathcal{N}_{a_1a}^{a_4}\mathcal{N}_{a_2a_3}^{a}} \langle w'_{(a_4)}, \mathcal{Y}_{3,j}^{a}(w_{(a_1)}, x_1)\mathcal{Y}_{4,j}^{a}(w_{(a_2)}, x_2)w_{(a_3)}\rangle_{W^{a_4}}\Big|_{x_1=z_1, x_2=z_2} \tag{2.29}$$

defined on the region $|z_1| > |z_2| > 0$ are equal on the intersection $|z_1| > |z_2| > |z_1 - z_2| > 0$.

Proof. By skew-symmetry, on the region $|z_2| > |z_1 - z_2| > 0$, we have

$$\begin{aligned}
&\langle w'_{(a_4)}, \mathcal{Y}_2(\mathcal{Y}_1(w_{(a_1)}, x_0)w_{(a_2)}, x_2)w_{(a_3)}\rangle_{W^{a_4}}\Big|_{x_0=z_1-z_2, x_2=z_2}\\
&\quad= \langle w'_{(a_4)}, \Omega^{-1}(\Omega(\mathcal{Y}_2))(\mathcal{Y}_1(w_{(a_1)}, x_0)w_{(a_2)}, x_2)w_{(a_3)}\rangle_{W^{a_4}}\Big|_{x_0=z_1-z_2, x_2=z_2}\\
&\quad= \langle w'_{(a_4)}, e^{x_2L(-1)}\Omega(\mathcal{Y}_2)(w_{(a_3)}, e^{\pi i}x_2)\mathcal{Y}_1(w_{(a_1)}, x_0)w_{(a_2)}\rangle_{W^{a_4}}\Big|_{x_0=z_1-z_2, x_2=z_2}\\
&\quad= \langle e^{x_2L(1)}w'_{(a_4)}, \Omega(\mathcal{Y}_2)(w_{(a_3)}, e^{\pi i}x_2)\mathcal{Y}_1(w_{(a_1)}, x_0)w_{(a_2)}\rangle_{W^{a_4}}\Big|_{x_0=z_1-z_2, x_2=z_2}.
\end{aligned} \tag{2.30}$$

Moreover, applying skew-symmetry again, we have

$$\begin{aligned}
&\langle e^{x_2L(1)}w'_{(a_4)}, \Omega(\mathcal{Y}_2)(w_{(a_3)}, e^{\pi i}x_2)\mathcal{Y}_1(w_{(a_1)}, x_0)w_{(a_2)}\rangle_{W^{a_4}}\Big|_{x_0=z_1-z_2, x_2=z_2}\\
&\quad= \langle e^{x_2L(1)}w'_{(a_4)}, \Omega(\mathcal{Y}_2)(w_{(a_3)}, e^{\pi i}x_2)\Omega^{-1}(\Omega(\mathcal{Y}_1))(w_{(a_1)}, x_0)w_{(a_2)}\rangle_{W^{a_4}}\Big|_{x_0=z_1-z_2, x_2=z_2}\\
&\quad= \langle e^{x_2L(1)}w'_{(a_4)}, \Omega(\mathcal{Y}_2)(w_{(a_3)}, e^{\pi i}x_2)e^{x_0L(-1)}\Omega(\mathcal{Y}_1)(w_{(a_2)}, e^{\pi i}x_0)w_{(a_1)}\rangle_{W^{a_4}}\Big|_{x_0=z_1-z_2, x_2=z_2}\\
&\quad= \langle e^{x_1L(1)}w'_{(a_4)}, e^{-x_0L(-1)}\Omega(\mathcal{Y}_2)(w_{(a_3)}, e^{\pi i}x_2)e^{x_0L(-1)}\Omega(\mathcal{Y}_1)(w_{(a_2)}, e^{\pi i}x_0)w_{(a_1)}\rangle_{W^{a_4}}\Big|_{\substack{x_0=z_1-z_2\\ x_1=z_1, x_2=z_2}}\\
&\quad= \langle e^{x_1L(1)}w'_{(a_4)}, \Omega(\mathcal{Y}_2)(w_{(a_3)}, e^{\pi i}x_1)\Omega(\mathcal{Y}_1)(w_{(a_2)}, e^{\pi i}x_0)w_{(a_1)}\rangle_{W^{a_4}}\Big|_{x_0=z_1-z_2, x_1=z_1, x_2=z_2}
\end{aligned} \tag{2.31}$$

on the region $|z_1| > |z_2| > |z_1 - z_2| > 0$. Then the associativity property implies that there exist $\mathcal{Y}_{5,j}^{a} \in \mathcal{V}_{a_3a_2}^{a}$ and $\mathcal{Y}_{6,j}^{a} \in \mathcal{V}_{aa_1}^{a_4}$ for $j = 1, \ldots, \mathcal{N}_{a_3a_2}^{a}\mathcal{N}_{aa_1}^{a_4}$ and $a \in \mathcal{A}$ such that the last line of (2.31) defined on the region $|z_1| > |z_1 - z_2| > 0$ and the (multivalued) analytic function

$$\sum_{a\in\mathcal{A}} \sum_{j=1}^{\mathcal{N}_{a_3a_2}^{a}\mathcal{N}_{aa_1}^{a_4}} \langle e^{x_1L(1)}w'_{(a_4)}, \mathcal{Y}_{6,j}^{a}(\mathcal{Y}_{5,j}^{a}(w_{(a_3)}, e^{\pi i}x_2)w_{(a_2)}, e^{\pi i}x_0)w_{(a_1)}\rangle_{W^{a_4}}\Big|_{\substack{x_0=z_1-z_2\\ x_1=z_1, x_2=z_2}} \tag{2.32}$$

defined on the region $|z_1 - z_2| > |z_2| > 0$ are equal on the intersection $|z_1| > |z_1 - z_2| > |z_2| > 0$. Moreover, by skew-symmetry, we have

$$\begin{aligned}
&\sum_{a\in\mathcal{A}} \sum_{j=1}^{\mathcal{N}_{a_3a_2}^{a}\mathcal{N}_{aa_1}^{a_4}} \langle e^{x_1L(1)}w'_{(a_4)}, \mathcal{Y}_{6,j}^{a}(\mathcal{Y}_{5,j}^{a}(w_{(a_3)}, e^{\pi i}x_2)w_{(a_2)}, e^{\pi i}x_0)w_{(a_1)}\rangle_{W^{a_4}}\big|_{x_0=z_1-z_2,x_1=z_1,x_2=z_2}\\
&\quad = \sum_{a\in\mathcal{A}} \sum_{j=1}^{\mathcal{N}_{a_3a_2}^{a}\mathcal{N}_{aa_1}^{a_4}} \langle e^{x_1L(1)}w'_{(a_4)}, \mathcal{Y}_{6,j}^{a}(\Omega(\Omega^{-1}(\mathcal{Y}_{5,j}^{a}))(w_{(a_3)}, e^{\pi i}x_2)w_{(a_2)}, e^{\pi i}x_0)w_{(a_1)}\rangle_{W^{a_4}}\Bigg|_{\substack{x_0=z_1-z_2\\ x_1=z_1\\ x_2=z_2}}\\
&\quad = \sum_{a\in\mathcal{A}} \sum_{j=1}^{\mathcal{N}_{a_3a_2}^{a}\mathcal{N}_{aa_1}^{a_4}} \langle e^{x_1L(1)}w'_{(a_4)}, \mathcal{Y}_{6,j}^{a}(e^{-x_2L(-1)}\Omega^{-1}(\mathcal{Y}_{5,j}^{a})(w_{(a_2)}, x_2)w_{(a_3)}, e^{\pi i}x_0)w_{(a_1)}\rangle_{W^{a_4}}\Bigg|_{\substack{x_0=z_1-z_2\\ x_1=z_1\\ x_2=z_2}}\\
&\quad = \sum_{a\in\mathcal{A}} \sum_{j=1}^{\mathcal{N}_{a_3a_2}^{a}\mathcal{N}_{aa_1}^{a_4}} \langle e^{x_1L(1)}w'_{(a_4)}, \mathcal{Y}_{6,j}^{a}(\Omega^{-1}(\mathcal{Y}_{5,j}^{a})(w_{(a_2)}, x_2)w_{(a_3)}, e^{\pi i}x_1)w_{(a_1)}\rangle_{W^{a_4}}\Bigg|_{\substack{x_1=z_1\\ x_2=z_2}}
\end{aligned}\tag{2.33}$$

on the region $|z_1| > |z_1 - z_2| > |z_2| > 0$. Applying skew-symmetry again, we further get

$$\begin{aligned}
&\sum_{a\in\mathcal{A}} \sum_{j=1}^{\mathcal{N}_{a_3a_2}^{a}\mathcal{N}_{aa_1}^{a_4}} \langle e^{x_1L(1)}w'_{(a_4)}, \mathcal{Y}_{6,j}^{a}(\Omega^{-1}(\mathcal{Y}_{5,j}^{a})(w_{(a_2)}, x_2)w_{(a_3)}, e^{\pi i}x_1)w_{(a_1)}\rangle_{W^{a_4}}\Bigg|_{\substack{x_1=z_1\\ x_2=z_2}}\\
&\quad = \sum_{a\in\mathcal{A}} \sum_{j=1}^{\mathcal{N}_{a_3a_2}^{a}\mathcal{N}_{aa_1}^{a_4}} \langle e^{x_1L(1)}w'_{(a_4)}, \Omega(\Omega^{-1}(\mathcal{Y}_{6,j}^{a}))(\Omega^{-1}(\mathcal{Y}_{5,j}^{a})(w_{(a_2)}, x_2)w_{(a_3)}, e^{\pi i}x_1)w_{(a_1)}\rangle_{W^{a_4}}\Bigg|_{\substack{x_1=z_1\\ x_2=z_2}}\\
&\quad = \sum_{a\in\mathcal{A}} \sum_{j=1}^{\mathcal{N}_{a_3a_2}^{a}\mathcal{N}_{aa_1}^{a_4}} \langle e^{x_1L(1)}w'_{(a_4)}, e^{-x_1L(-1)}\Omega^{-1}(\mathcal{Y}_{6,j}^{a})(w_{(a_1)}, x_1)\Omega^{-1}(\mathcal{Y}_{5,j}^{a})(w_{(a_2)}, x_2)w_{(a_3)}\rangle_{W^{a_4}}\Bigg|_{\substack{x_1=z_1\\ x_2=z_2}}\\
&\quad = \sum_{a\in\mathcal{A}} \sum_{j=1}^{\mathcal{N}_{a_1a}^{a_4}\mathcal{N}_{a_2a_3}^{a}} \langle w'_{(a_4)}, \Omega^{-1}(\mathcal{Y}_{6,j}^{a})(w_{(a_1)}, x_1)\Omega^{-1}(\mathcal{Y}_{5,j}^{a})(w_{(a_2)}, x_2)w_{(a_3)}\rangle_{W^{a_4}}\Bigg|_{x_1=z_1,x_2=z_2}
\end{aligned}\tag{2.34}$$

on the region $|z_1| > |z_2| > 0$, where $\mathcal{N}_{a_1a}^{a_4} = \mathcal{N}_{aa_1}^{a_4}$, $\mathcal{N}_{a_2a_3}^{a} = \mathcal{N}_{a_3a_2}^{a}$. Let $\mathcal{Y}_{3,j}^{a} = \Omega^{-1}(\mathcal{Y}_{6,j}^{a})$ and $\mathcal{Y}_{4,j}^{a} = \Omega^{-1}(\mathcal{Y}_{5,j}^{a})$ for $j = 1, \dots, \mathcal{N}_{a_1a}^{a_4}\mathcal{N}_{a_2a_3}^{a}$ and $a \in \mathcal{A}$. Then we can obtain that, for any $w_{(a_i)} \in W^{a_i}$, $i = 1, 2, 3$, and $w'_{(a_4)} \in (W^{a_4})'$, the (multivalued) analytic function

$$\langle w'_{(a_4)}, \mathcal{Y}_2(\mathcal{Y}_1(w_{(a_1)}, x_0)w_{(a_2)}, x_2)w_{(a_3)}\rangle_{W^{a_4}}\big|_{x_0=z_1-z_2,x_2=z_2} \tag{2.35}$$

defined on the region $|z_2| > |z_1 - z_2| > 0$ and the (multivalued) analytic function

$$\sum_{a\in\mathcal{A}} \sum_{j=1}^{\mathcal{N}_{a_1a}^{a_4}\mathcal{N}_{a_2a_3}^{a}} \langle w'_{(a_4)}, \mathcal{Y}_{3,j}^{a}(w_{(a_1)}, x_1)\mathcal{Y}_{4,j}^{a}(w_{(a_2)}, x_2)w_{(a_3)}\rangle_{W^{a_4}}\Bigg|_{x_1=z_1,x_2=z_2} \tag{2.36}$$

defined on the region $|z_1| > |z_2| > 0$ are equal on the intersection $|z_1| > |z_2| > |z_1 - z_2| > 0$. So the second associativity holds. ∎

Proposition 2.7 (Commutativity). *Let $(W, \mathcal{A}, \{\mathcal{V}_{a_1a_2}^{a_3}\}, \mathbf{1}, \omega)$ be an intertwining operator algebra. Then we have the following commutativity: For any $a_1, a_2, a_3, a_4, a_5 \in \mathcal{A}$, $\mathcal{Y}_1 \in \mathcal{V}_{a_1a_5}^{a_4}$ and $\mathcal{Y}_2 \in \mathcal{V}_{a_2a_3}^{a_5}$, there exist $\mathcal{Y}_{5,j}^{a} \in \mathcal{V}_{a_2a}^{a_4}$ and $\mathcal{Y}_{6,j}^{a} \in \mathcal{V}_{a_1a_3}^{a}$ for $j = 1, \ldots, \mathcal{N}_{a_2a}^{a_4}\mathcal{N}_{a_1a_3}^{a}$ and $a \in \mathcal{A}$, such that for $w_{(a_i)} \in W^{a_i}$, $i = 1, 2, 3$, and $w'_{(a_4)} \in (W^{a_4})'$, the (multivalued) analytic function*

$$\langle w'_{(a_4)}, \mathcal{Y}_1(w_{(a_1)}, x_1)\mathcal{Y}_2(w_{(a_2)}, x_2)w_{(a_3)}\rangle_{W^{a_4}}\big|_{x_1=z_1, x_2=z_2} \tag{2.37}$$

defined on the region $|z_1| > |z_2| > 0$ and the (multivalued) analytic function

$$\sum_{a\in\mathcal{A}}\sum_{j=1}^{\mathcal{N}_{a_2a}^{a_4}\mathcal{N}_{a_1a_3}^{a}} \langle w'_{(a_4)}, \mathcal{Y}_{5,j}^{a}(w_{(a_2)}, x_2)\mathcal{Y}_{6,j}^{a}(w_{(a_1)}, x_1)w_{(a_3)}\rangle_{W^{a_4}}\Bigg|_{x_1=z_1, x_2=z_2} \tag{2.38}$$

defined on the region $|z_2| > |z_1| > 0$ are analytic extensions of each other.

Proof. By the associativity property, we know that, for $\mathcal{Y}_1 \in \mathcal{V}_{a_1a_5}^{a_4}$ and $\mathcal{Y}_2 \in \mathcal{V}_{a_2a_3}^{a_5}$, there exist $\mathcal{Y}_{3,j}^{a} \in \mathcal{V}_{a_1a_2}^{a}$ and $\mathcal{Y}_{4,j}^{a} \in \mathcal{V}_{aa_3}^{a_4}$ for $j = 1, \ldots, \mathcal{N}_{a_1a_2}^{a}\mathcal{N}_{aa_3}^{a_4}$ and $a \in \mathcal{A}$, such that for $w_{(a_i)} \in W^{a_i}$, $i = 1, 2, 3$, and $w'_{(a_4)} \in (W^{a_4})'$, the (multivalued) analytic function

$$\langle w'_{(a_4)}, \mathcal{Y}_1(w_{(a_1)}, x_1)\mathcal{Y}_2(w_{(a_2)}, x_2)w_{(a_3)}\rangle_{W^{a_4}}\big|_{x_1=z_1, x_2=z_2} \tag{2.39}$$

defined on the region $|z_1| > |z_2| > 0$ and the (multivalued) analytic function

$$\sum_{a\in\mathcal{A}}\sum_{j=1}^{\mathcal{N}_{a_1a_2}^{a}\mathcal{N}_{aa_3}^{a_4}} \langle w'_{(a_4)}, \mathcal{Y}_{4,j}^{a}(\mathcal{Y}_{3,j}^{a}(w_{(a_1)}, x_0)w_{(a_2)}, x_2)w_{(a_3)}\rangle_{W^{a_4}}\Bigg|_{x_0=z_1-z_2, x_2=z_2} \tag{2.40}$$

defined on the region $|z_2| > |z_1 - z_2| > 0$ are equal on the intersection $|z_1| > |z_2| > |z_1 - z_2| > 0$. Applying skew-symmetry, we have

$$\begin{aligned}
&\sum_{a\in\mathcal{A}}\sum_{j=1}^{\mathcal{N}_{a_1a_2}^{a}\mathcal{N}_{aa_3}^{a_4}} \langle w'_{(a_4)}, \mathcal{Y}_{4,j}^{a}(\mathcal{Y}_{3,j}^{a}(w_{(a_1)}, x_0)w_{(a_2)}, x_2)w_{(a_3)}\rangle_{W^{a_4}}\Bigg|_{\substack{x_0=z_1-z_2\\ x_2=z_2}}\\
&\quad= \sum_{a\in\mathcal{A}}\sum_{j=1}^{\mathcal{N}_{a_1a_2}^{a}\mathcal{N}_{aa_3}^{a_4}} \langle w'_{(a_4)}, \mathcal{Y}_{4,j}^{a}(\Omega^{-1}(\Omega(\mathcal{Y}_{3,j}^{a}))(w_{(a_1)}, x_0)w_{(a_2)}, x_2)w_{(a_3)}\rangle_{W^{a_4}}\Bigg|_{\substack{x_0=z_1-z_2\\ x_2=z_2}}\\
&\quad= \sum_{a\in\mathcal{A}}\sum_{j=1}^{\mathcal{N}_{a_1a_2}^{a}\mathcal{N}_{aa_3}^{a_4}} \langle w'_{(a_4)}, \mathcal{Y}_{4,j}^{a}(e^{x_0L(-1)}\Omega(\mathcal{Y}_{3,j}^{a})(w_{(a_2)}, e^{\pi i}x_0)w_{(a_1)}, x_2)w_{(a_3)}\rangle_{W^{a_4}}\Bigg|_{\substack{x_0=z_1-z_2\\ x_2=z_2}}\\
&\quad= \sum_{a\in\mathcal{A}}\sum_{j=1}^{\mathcal{N}_{a_1a_2}^{a}\mathcal{N}_{aa_3}^{a_4}} \langle w'_{(a_4)}, \mathcal{Y}_{4,j}^{a}(\Omega(\mathcal{Y}_{3,j}^{a})(w_{(a_2)}, e^{\pi i}x_0)w_{(a_1)}, x_1)w_{(a_3)}\rangle_{W^{a_4}}\Bigg|_{\substack{x_0=z_1-z_2\\ x_1=z_1}}
\end{aligned} \tag{2.41}$$

on the region $|z_1| > |z_2| > |z_1 - z_2| > 0$. By the second associativity, there exist $\mathcal{Y}^a_{5,j} \in \mathcal{V}^{a_4}_{a_2 a}$ and $\mathcal{Y}^a_{6,j} \in \mathcal{V}^a_{a_1 a_3}$ for $j = 1, \dots, \mathcal{N}^{a_4}_{a_2 a}\mathcal{N}^a_{a_1 a_3}$ and $a \in \mathcal{A}$, such that the last line of (2.41) defined on the region $|z_1| > |z_1 - z_2| > 0$ and the (multivalued) analytic function

$$\sum_{a\in\mathcal{A}} \sum_{j=1}^{\mathcal{N}^{a_4}_{a_2 a}\mathcal{N}^a_{a_1 a_3}} \langle w'_{(a_4)}, \mathcal{Y}^a_{5,j}(w_{(a_2)}, x_2)\mathcal{Y}^a_{6,j}(w_{(a_1)}, x_1)w_{(a_3)}\rangle_{W^{a_4}}\Big|_{x_1=z_1, x_2=z_2} \tag{2.42}$$

defined on the region $|z_2| > |z_1| > 0$ are equal on the intersection $|z_2| > |z_1| > |z_1 - z_2| > 0$.

So the (multivalued) analytic function

$$\langle w'_{(a_4)}, \mathcal{Y}_1(w_{(a_1)}, x_1)\mathcal{Y}_2(w_{(a_2)}, x_2)w_{(a_3)}\rangle_{W^{a_4}}|_{x_1=z_1, x_2=z_2} \tag{2.43}$$

defined on the region $|z_1| > |z_2| > 0$ and the (multivalued) analytic function

$$\sum_{a\in\mathcal{A}} \sum_{j=1}^{\mathcal{N}^{a_4}_{a_2 a}\mathcal{N}^a_{a_1 a_3}} \langle w'_{(a_4)}, \mathcal{Y}^a_{5,j}(w_{(a_2)}, x_2)\mathcal{Y}^a_{6,j}(w_{(a_1)}, x_1)w_{(a_3)}\rangle_{W^{a_4}}\Big|_{x_1=z_1, x_2=z_2} \tag{2.44}$$

defined on the region $|z_2| > |z_1| > 0$ are analytic extensions of each other for any $w_{(a_i)} \in W^{a_i}$, $i = 1, 2, 3$, and $w'_{(a_4)} \in (W^{a_4})'$. So the commutativity holds. ∎

More relations among associativity, skew-symmetry and commutativity will be derived in the next section.

Now we give the preliminaries about the Jacobi identity. First, we need to discuss certain special multivalued analytic functions. Consider the simply connected regions in $\mathbb{C}^2$ obtained by cutting the regions $|z_1| > |z_2| > 0$ and $|z_2| > |z_1| > 0$ along the intersections of these regions with $\{(z_1, z_2) \in \mathbb{C}^2 \mid z_1 \in [0, +\infty)\} \cup \{(z_1, z_2) \in \mathbb{C}^2 \mid z_2 \in [0, +\infty)\}$, by cutting the region $|z_2| > |z_1 - z_2| > 0$ along the intersection of this region with $\{(z_1, z_2) \in \mathbb{C}^2 \mid z_2 \in [0, +\infty)\} \cup \{(z_1, z_2) \in \mathbb{C}^2 \mid z_1 - z_2 \in [0, +\infty)\}$, and by cutting the region $|z_1| > |z_1 - z_2| > 0$ along the intersection of this region with $\{(z_1, z_2) \in \mathbb{C}^2 \mid z_1 \in [0, +\infty)\} \cup \{(z_1, z_2) \in \mathbb{C}^2 \mid z_2 - z_1 \in [0, +\infty)\}$. We denote them by R_1, R_2, R_3 and R_4, respectively. For $a_1, a_2, a_3, a_4 \in \mathcal{A}$, let $\mathbb{G}^{a_1,a_2,a_3,a_4}$ be the set of multivalued analytic functions on

$$M^2 = \{(z_1, z_2) \in \mathbb{C}^2 \mid z_1, z_2 \neq 0, z_1 \neq z_2\} \tag{2.45}$$

with a choice of a single-valued branch on the region R_1 satisfying the following property: On the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$ and $|z_2| > |z_1 - z_2| > 0$, any branch of $f(z_1, z_2) \in \mathbb{G}^{a_1,a_2,a_3,a_4}$ can be expanded as

$$\sum_{a\in\mathcal{A}} z_1^{h_{a_4} - h_{a_1} - h_a} z_2^{h_a - h_{a_2} - h_{a_3}} F_a(z_1, z_2), \tag{2.46}$$

$$\sum_{a\in\mathcal{A}} z_2^{h_{a_4} - h_{a_2} - h_a} z_1^{h_a - h_{a_1} - h_{a_3}} G_a(z_1, z_2) \tag{2.47}$$

and

$$\sum_{a\in\mathcal{A}} z_2^{h_{a_4}-h_a-h_{a_3}}(z_1-z_2)^{h_a-h_{a_1}-h_{a_2}}H_a(z_1,z_2), \tag{2.48}$$

respectively, where for $a\in\mathcal{A}$,

$$F_a(z_1,z_2)\in\mathbb{C}[[z_2/z_1]][z_1,z_1^{-1},z_2,z_2^{-1}], \tag{2.49}$$

$$G_a(z_1,z_2)\in\mathbb{C}[[z_1/z_2]][z_1,z_1^{-1},z_2,z_2^{-1}] \tag{2.50}$$

and

$$H_a(z_1,z_2)\in\mathbb{C}[[(z_1-z_2)/z_2]][z_2,z_2^{-1},z_1-z_2,(z_1-z_2)^{-1}]. \tag{2.51}$$

We call the chosen single-valued branch on R_1 of an element of $\mathbb{G}^{a_1,a_2,a_3,a_4}$ the *preferred branch on* R_1. Consider the nonempty simply connected regions

$$S_1=\{(z_1,z_2)\in\mathbb{C}^2 \mid \mathrm{Re}z_1>\mathrm{Re}z_2>\mathrm{Re}(z_1-z_2)>0,\ \mathrm{Im}z_1>\mathrm{Im}z_2>\mathrm{Im}(z_1-z_2)>0\}$$

and

$$S_2=\{(z_1,z_2)\in\mathbb{C}^2 \mid \mathrm{Re}z_2>\mathrm{Re}z_1>\mathrm{Re}(z_2-z_1)>0,\ \mathrm{Im}z_2>\mathrm{Im}z_1>\mathrm{Im}(z_2-z_1)>0\}.$$

Then the restriction of the preferred branch on R_1 of an element of $\mathbb{G}^{a_1,a_2,a_3,a_4}$ to the region $S_1\subset R_1\cap R_3$ gives a single-valued branch of the element on R_3. We call this branch the *preferred branch on* R_3. Similarly, the restriction of the preferred branch on R_1 to the region $S_1\subset R_1\cap R_4$ gives a single-valued branch of the element on R_4 and we call this branch the *preferred branch on* R_4. Moreover, the restriction of the preferred branch on R_4 to the region $S_2\subset R_4\cap R_2$ gives a single-valued branch of the element on R_2 and we call this branch the *preferred branch on* R_2.

Given two elements of $\mathbb{G}^{a_1,a_2,a_3,a_4}$, the addition of their preferred branches is also a single-valued branch of a multivalued analytic function on M^2. This multivalued analytic function on M^2 together with the addition of the preferred branches is also an element of $\mathbb{G}^{a_1,a_2,a_3,a_4}$. We define this element as the addition of the two elements of $\mathbb{G}^{a_1,a_2,a_3,a_4}$. Thus we obtain an addition operation in $\mathbb{G}^{a_1,a_2,a_3,a_4}$. Similarly we have a scalar multiplication in $\mathbb{G}^{a_1,a_2,a_3,a_4}$. It is clear that $\mathbb{G}^{a_1,a_2,a_3,a_4}$ with these operations is a vector space.

Given an element of $\mathbb{G}^{a_1,a_2,a_3,a_4}$, the preferred branches of this function on R_1, R_2 and R_3 give formal series in

$$\coprod_{a\in\mathcal{A}} x_1^{h_{a_4}-h_{a_1}-h_a}x_2^{h_a-h_{a_2}-h_{a_3}}\mathbb{C}[[x_2/x_1]][x_1,x_1^{-1},x_2,x_2^{-1}], \tag{2.52}$$

$$\coprod_{a\in\mathcal{A}} x_2^{h_{a_4}-h_{a_2}-h_a}x_1^{h_a-h_{a_1}-h_{a_3}}\mathbb{C}[[x_1/x_2]][x_1,x_1^{-1},x_2,x_2^{-1}] \tag{2.53}$$

and

$$\coprod_{a\in\mathcal{A}} x_2^{h_{a_4}-h_a-h_{a_3}} x_0^{h_a-h_{a_1}-h_{a_2}}\mathbb{C}[[x_0/x_2]][x_0,x_0^{-1},x_2,x_2^{-1}], \tag{2.54}$$

respectively. Thus we have linear maps

$$\begin{aligned}
\iota_{12}: \mathbb{G}^{a_1,a_2,a_3,a_4} &\to \coprod_{a\in\mathcal{A}} x_1^{h_{a_4}-h_{a_1}-h_a} x_2^{h_a-h_{a_2}-h_{a_3}}\mathbb{C}[[x_2/x_1]][x_1,x_1^{-1},x_2,x_2^{-1}] &(2.55)\\
\iota_{21}: \mathbb{G}^{a_1,a_2,a_3,a_4} &\to \coprod_{a\in\mathcal{A}} x_2^{h_{a_4}-h_{a_2}-h_a} x_1^{h_a-h_{a_1}-h_{a_3}}\mathbb{C}[[x_1/x_2]][x_1,x_1^{-1},x_2,x_2^{-1}] &(2.56)\\
\iota_{20}: \mathbb{G}^{a_1,a_2,a_3,a_4} &\to \coprod_{a\in\mathcal{A}} x_2^{h_{a_4}-h_a-h_{a_3}} x_0^{h_a-h_{a_1}-h_{a_2}}\mathbb{C}[[x_0/x_2]][x_0,x_0^{-1},x_2,x_2^{-1}] &(2.57)
\end{aligned}$$

generalizing ι_{12}, ι_{21} and ι_{20} discussed before. Since analytic extensions are unique, these maps are injective.

For any $a_1,a_2,a_3,a_4\in\mathcal{A}$, $\mathbb{G}^{a_1,a_2,a_3,a_4}$ is a module over the ring

$$\mathbb{C}[x_1,x_1^{-1},x_2,x_2^{-1},(x_1-x_2)^{-1}]. \tag{2.58}$$

We have the following lemma proved in [H9]:

Lemma 2.8. *For any $a_1,a_2,a_3,a_4\in\mathcal{A}$, the module $\mathbb{G}^{a_1,a_2,a_3,a_4}$ is free.*

For convenience of the rest of the paper, we fix a basis $\{e_\alpha^{a_1,a_2,a_3,a_4}\}_{\alpha\in\mathbb{A}(a_1,a_2,a_3,a_4)}$ of the free module $\mathbb{G}^{a_1,a_2,a_3,a_4}$ for any $a_1,a_2,a_3,a_4\in\mathcal{A}$.

Next, we shall define two maps, which correspond to the multiplication and iterates of intertwining operators, respectively. The first one is

$$\begin{aligned}
\mathbf{P}: \coprod_{a_1,a_2,a_3,a_4,a_5\in\mathcal{A}} \mathcal{V}_{a_1a_5}^{a_4}\otimes\mathcal{V}_{a_2a_3}^{a_5} &\to (\mathrm{Hom}(W\otimes W\otimes W,W))\{x_1,x_2\}\\
\mathcal{Z} &\mapsto \mathbf{P}(\mathcal{Z})
\end{aligned} \tag{2.59}$$

defined using products of intertwining operators as follows: For

$$\mathcal{Z}\in \coprod_{a_1,a_2,a_3,a_4,a_5\in\mathcal{A}} \mathcal{V}_{a_1a_5}^{a_4}\otimes\mathcal{V}_{a_2a_3}^{a_5}, \tag{2.60}$$

the element $\mathbf{P}(\mathcal{Z})$ to be defined can also be viewed as a linear map from $W\otimes W\otimes W$ to $W\{x_1,x_2\}$. For any $w_1,w_2,w_3\in W$, we denote the image of $w_1\otimes w_2\otimes w_3$ under this map by $(\mathbf{P}(\mathcal{Z}))(w_1,w_2,w_3;x_1,x_2)$. Then we define $\mathbf{P}$ by linearity and by

$$\begin{aligned}
&(\mathbf{P}(\mathcal{Y}_1\otimes\mathcal{Y}_2))(w_{(a_6)},w_{(a_7)},w_{(a_8)};x_1,x_2)\\
&\quad=\begin{cases}\mathcal{Y}_1(w_{(a_6)},x_1)\mathcal{Y}_2(w_{(a_7)},x_2)w_{(a_8)}, & a_6=a_1,a_7=a_2,a_8=a_3,\\ 0, & \text{otherwise}\end{cases}
\end{aligned} \tag{2.61}$$

for $a_1, \dots, a_8 \in \mathcal{A}$, $\mathcal{Y}_1 \in \mathcal{V}_{a_1 a_5}^{a_4}$, $\mathcal{Y}_2 \in \mathcal{V}_{a_2 a_3}^{a_5}$, and $w_{(a_6)} \in W^{a_6}$, $w_{(a_7)} \in W^{a_7}$, $w_{(a_8)} \in W^{a_8}$. Then we have an isomorphism

$$\tilde{\mathbf{P}} : \quad \frac{\coprod_{a_1,a_2,a_3,a_4,a_5 \in \mathcal{A}} \mathcal{V}_{a_1 a_5}^{a_4} \otimes \mathcal{V}_{a_2 a_3}^{a_5}}{\mathrm{Ker}\ \mathbf{P}} \longrightarrow \mathbf{P}\left(\coprod_{a_1,a_2,a_3,a_4,a_5 \in \mathcal{A}} \mathcal{V}_{a_1 a_5}^{a_4} \otimes \mathcal{V}_{a_2 a_3}^{a_5} \right) \tag{2.62}$$

which makes the following diagram commute:

$$\begin{tikzcd} \coprod_{a_1,a_2,a_3,a_4,a_5 \in \mathcal{A}} \mathcal{V}_{a_1 a_5}^{a_4} \otimes \mathcal{V}_{a_2 a_3}^{a_5} \arrow[r, "\mathbf{P}"] \arrow[d, "\pi_P"'] & \mathbf{P}\left(\coprod_{a_1,a_2,a_3,a_4,a_5 \in \mathcal{A}} \mathcal{V}_{a_1 a_5}^{a_4} \otimes \mathcal{V}_{a_2 a_3}^{a_5} \right), \\ \frac{\coprod_{a_1,a_2,a_3,a_4,a_5 \in \mathcal{A}} \mathcal{V}_{a_1 a_5}^{a_4} \otimes \mathcal{V}_{a_2 a_3}^{a_5}}{\mathrm{Ker}\ \mathbf{P}} \arrow[ur, "\tilde{\mathbf{P}}"] & \end{tikzcd} \tag{2.63}$$

where π_P is the corresponding canonical projective map. When there is no ambiguity, we shall denote $\pi_P(\mathcal{Z})$ by $[\mathcal{Z}]_P$ or by $\mathcal{Z} + \mathrm{Ker}\ \mathbf{P}$ for $\mathcal{Z} \in \coprod_{a_1,a_2,a_3,a_4,a_5 \in \mathcal{A}} \mathcal{V}_{a_1 a_5}^{a_4} \otimes \mathcal{V}_{a_2 a_3}^{a_5}$.

The second map is

$$\begin{aligned} \mathbf{I} : \coprod_{a_1,a_2,a_3,a_4,a_5 \in \mathcal{A}} \mathcal{V}_{a_1 a_2}^{a_5} \otimes \mathcal{V}_{a_5 a_3}^{a_4} &\to (\mathrm{Hom}(W \otimes W \otimes W, W))\{x_0, x_2\} \\ \mathcal{Z} &\mapsto \mathbf{I}(\mathcal{Z}) \end{aligned} \tag{2.64}$$

defined similarly using iterates of intertwining operators as follows: For

$$\mathcal{Z} \in \coprod_{a_1,a_2,a_3,a_4,a_5 \in \mathcal{A}} \mathcal{V}_{a_1 a_2}^{a_5} \otimes \mathcal{V}_{a_5 a_3}^{a_4}, \tag{2.65}$$

the element $\mathbf{I}(\mathcal{Z})$ to be defined can also be viewed as a linear map from $W \otimes W \otimes W$ to $W\{x_0, x_2\}$. For any $w_1, w_2, w_3 \in W$, we denote the image of $w_1 \otimes w_2 \otimes w_3$ under this map by $(\mathbf{I}(\mathcal{Z}))(w_1, w_2, w_3; x_0, x_2)$. We define $\mathbf{I}$ by linearity and by

$$\begin{aligned} &(\mathbf{I}(\mathcal{Y}_1 \otimes \mathcal{Y}_2))(w_{(a_6)}, w_{(a_7)}, w_{(a_8)}; x_0, x_2) \\ &\quad = \begin{cases} \mathcal{Y}_2(\mathcal{Y}_1(w_{(a_6)}, x_0) w_{(a_7)}, x_2) w_{(a_8)}, & a_6 = a_1, a_7 = a_2, a_8 = a_3, \\ 0, & \text{otherwise} \end{cases} \end{aligned} \tag{2.66}$$

for $a_1, \dots, a_8 \in \mathcal{A}$, $\mathcal{Y}_1 \in \mathcal{V}_{a_1 a_2}^{a_5}$, $\mathcal{Y}_2 \in \mathcal{V}_{a_5 a_3}^{a_4}$, and $w_{(a_6)} \in W^{a_6}$, $w_{(a_7)} \in W^{a_7}$, $w_{(a_8)} \in W^{a_8}$. Then we have an isomorphism

$$\tilde{\mathbf{I}} : \quad \frac{\coprod_{a_1,a_2,a_3,a_4,a_5 \in \mathcal{A}} \mathcal{V}_{a_1 a_2}^{a_5} \otimes \mathcal{V}_{a_5 a_3}^{a_4}}{\mathrm{Ker}\ \mathbf{I}} \longrightarrow \mathbf{I}\left(\coprod_{a_1,a_2,a_3,a_4,a_5 \in \mathcal{A}} \mathcal{V}_{a_1 a_2}^{a_5} \otimes \mathcal{V}_{a_5 a_3}^{a_4} \right) \tag{2.67}$$

which makes the following diagram commute:

$$\begin{tikzcd}
\coprod_{a_1,a_2,a_3,a_4,a_5\in\mathcal{A}} \mathcal{V}_{a_1a_2}^{a_5}\otimes\mathcal{V}_{a_5a_3}^{a_4} \arrow[r, "\mathbf{I}"] \arrow[d, "\pi_I"'] & \mathbf{I}\left(\coprod_{a_1,a_2,a_3,a_4,a_5\in\mathcal{A}} \mathcal{V}_{a_1a_2}^{a_5}\otimes\mathcal{V}_{a_5a_3}^{a_4}\right), \\
\dfrac{\coprod_{a_1,a_2,a_3,a_4,a_5\in\mathcal{A}} \mathcal{V}_{a_1a_2}^{a_5}\otimes\mathcal{V}_{a_5a_3}^{a_4}}{\mathrm{Ker}\,\mathbf{I}} \arrow[ur, "\tilde{\mathbf{I}}"] &
\end{tikzcd} \tag{2.68}$$

where π_I is the corresponding canonical projective map. When there is no ambiguity, we shall denote $\pi_I(\mathcal{Z})$ by $[\mathcal{Z}]_I$ or by $\mathcal{Z}+\mathrm{Ker}\,\mathbf{I}$ for $\mathcal{Z}\in\coprod_{a_1,a_2,a_3,a_4,a_5\in\mathcal{A}}\mathcal{V}_{a_1a_2}^{a_5}\otimes\mathcal{V}_{a_5a_3}^{a_4}$.

We shall call $\mathbf{P}$ and $\mathbf{I}$ the *multiplication of intertwining operators* and the *iterates of intertwining operators*, respectively.

Then we shall derive some isomorphisms from the associativity and the skew-symmetry properties.

Note that in the associativity property, $\sum_{a\in\mathcal{A}}\sum_{i=1}^{\mathcal{N}_{a_1a_2}^{a}\mathcal{N}_{aa_3}^{a_4}}\mathcal{Y}_{3,i}^{a}\otimes\mathcal{Y}_{4,i}^{a}$ may not be unique. But we have the following result:

Lemma 2.9. *For $a_1,\cdots,a_5\in\mathcal{A}$, $\mathcal{Y}_1\in\mathcal{V}_{a_1a_5}^{a_4}$ and $\mathcal{Y}_2\in\mathcal{V}_{a_2a_3}^{a_5}$:*

1. *There exist $\mathcal{Y}_{3,i}^{a}\in\mathcal{V}_{a_1a_2}^{a}$ and $\mathcal{Y}_{4,i}^{a}\in\mathcal{V}_{aa_3}^{a_4}$ for $i=1,\ldots,\mathcal{N}_{a_1a_2}^{a}\mathcal{N}_{aa_3}^{a_4}$, $a\in\mathcal{A}$, such that for any $w_1,w_2,w_3\in W$ and $w'\in W'$,*

$$\langle w',(\mathbf{P}(\mathcal{Y}_1\otimes\mathcal{Y}_2))(w_1,w_2,w_3;x_1,x_2)\rangle_W\big|_{x_1^n=e^{n\log z_1},x_2^n=e^{n\log z_2}} \tag{2.69}$$

is equal to

$$\sum_{a\in\mathcal{A}}\sum_{i=1}^{\mathcal{N}_{a_1a_2}^{a}\mathcal{N}_{aa_3}^{a_4}}\langle w',(\mathbf{I}(\mathcal{Y}_{3,i}^{a}\otimes\mathcal{Y}_{4,i}^{a}))(w_1,w_2,w_3;x_0,x_2)\rangle_W\Big|_{x_0^n=e^{n\log(z_1-z_2)},x_2^n=e^{n\log z_2}} \tag{2.70}$$

on the region

$$S_1=\{(z_1,z_2)\in\mathbb{C}^2\mid \mathrm{Re}z_1>\mathrm{Re}z_2>\mathrm{Re}(z_1-z_2)>0,\ \mathrm{Im}z_1>\mathrm{Im}z_2>\mathrm{Im}(z_1-z_2)>0\}.$$

2. *Assume that $\{\tilde{\mathcal{Y}}_{3,i}^{a},\tilde{\mathcal{Y}}_{4,i}^{a}\mid i=1,\ldots,\mathcal{N}_{a_1a_2}^{a}\mathcal{N}_{aa_3}^{a_4},a\in\mathcal{A}\}$ is another set of intertwining operators satisfying Conclusion 1, then we have*

$$\sum_{a\in\mathcal{A}}\sum_{i=1}^{\mathcal{N}_{a_1a_2}^{a}\mathcal{N}_{aa_3}^{a_4}}\tilde{\mathcal{Y}}_{3,i}^{a}\otimes\tilde{\mathcal{Y}}_{4,i}^{a}\in\sum_{a\in\mathcal{A}}\sum_{i=1}^{\mathcal{N}_{a_1a_2}^{a}\mathcal{N}_{aa_3}^{a_4}}\mathcal{Y}_{3,i}^{a}\otimes\mathcal{Y}_{4,i}^{a}+\mathrm{Ker}\,\mathbf{I}. \tag{2.71}$$

Proof. By associativity, there exist $\mathcal{Y}^a_{3,i} \in \mathcal{V}^a_{a_1a_2}$ and $\mathcal{Y}^a_{4,i} \in \mathcal{V}^{a_4}_{aa_3}$ for $i = 1, \dots, \mathcal{N}^a_{a_1a_2}\mathcal{N}^{a_4}_{aa_3}$, $a \in \mathcal{A}$, such that for $w_{(a_j)} \in W^{a_j}$, $j = 1, 2, 3$, and $w'_{(a_4)} \in (W^{a_4})'$, the multivalued analytic functions (2.24) and (2.25) are equal on the region $|z_1| > |z_2| > |z_1 - z_2| > 0$. So on the simply connected open subset S_1 of this region,

$$\langle w'_{(a_4)}, \mathcal{Y}_1(w_{(a_1)}, x_1)\mathcal{Y}_2(w_{(a_2)}, x_2)w_{(a_3)}\rangle_{W^{a_4}}\big|_{x_1^n = e^{n \log z_1}, x_2^n = e^{n \log z_2}} \tag{2.72}$$

as a particular single-valued branch of (2.24) is equal to a single-valued branch of (2.25) on S_1. By definition, any single-valued branch of (2.25) on S_1 is of the form

$$\sum_{a \in \mathcal{A}} \sum_{i=1}^{\mathcal{N}^a_{a_1a_2}\mathcal{N}^{a_4}_{aa_3}} \langle w'_{(a_4)}, \mathcal{Y}^a_{4,i}(\mathcal{Y}^a_{3,i}(w_{(a_1)}, x_0)w_{(a_2)}, x_2)w_{(a_3)}\rangle_{W^{a_4}}\Bigg|_{\substack{x_0^n = e^{n(\log(z_1 - z_2) + 2k\pi i)} \\ x_2^n = e^{n(\log z_2 + 2l\pi i)}}} \tag{2.73}$$

for some $k, l \in \mathbb{Z}$. We also know that for any modules W_1, W_2, W_3, any intertwining operator

$$\begin{aligned} \mathcal{Y} : W_1 \otimes W_2 &\to W_3\{x\} \\ w_1 \otimes w_2 &\mapsto \mathcal{Y}(w_1, x)w_2 = \sum_{n \in \mathbb{C}} \mathcal{Y}_{(n)}(w_1)w_2x^{-n-1} \end{aligned} \tag{2.74}$$

of type $\binom{W_3}{W_1W_2}$ and any $p \in \mathbb{Z}$, the map from $W_1 \otimes W_2$ to $W_3\{x\}$ given by

$$w_1 \otimes w_2 \mapsto \sum_{n \in \mathbb{C}} \mathcal{Y}_{(n)}(w_1)w_2e^{2\pi p(-n-1)i}x^{-n-1} \tag{2.75}$$

for $w_1 \in W_1$ and $w_2 \in W_2$ is also an intertwining operator of the same type. Moreover, by weight condition in Definition 2.3, we know that for any $a, b, c \in \mathcal{A}$ and any $\mathcal{Y} \in \mathcal{V}^c_{ab}$, we may refine (2.74):

$$\begin{aligned} \mathcal{Y} : W^a \otimes W^b &\to W^c\{x\} \\ w_{(a)} \otimes w_{(b)} &\mapsto \mathcal{Y}(w_{(a)}, x)w_{(b)} = \sum_{n \in \mathbb{C}} \mathcal{Y}_{(n)}(w_{(a)})w_{(b)}x^{-n-1} \\ &\qquad = \sum_{n \in s + \mathbb{Z}} \mathcal{Y}_{(n)}(w_{(a)})w_{(b)}x^{-n-1} \end{aligned} \tag{2.76}$$

with $s = h_a + h_b - h_c \in \mathbb{R}$. So for any $p \in \mathbb{Z}$, the map from $W^a \otimes W^b$ to $W^c\{x\}$ given by

$$\begin{aligned} w_{(a)} \otimes w_{(b)} \mapsto\ & \sum_{n \in \mathbb{C}} \mathcal{Y}_{(n)}(w_{(a)})w_{(b)}e^{2\pi p(-n-1)i}x^{-n-1} \\ &= \sum_{n \in s + \mathbb{Z}} \mathcal{Y}_{(n)}(w_{(a)})w_{(b)}e^{2\pi p(-n-1)i}x^{-n-1} \\ &= e^{-2\pi psi} \sum_{n \in s + \mathbb{Z}} \mathcal{Y}_{(n)}(w_{(a)})w_{(b)}x^{-n-1} \\ &= e^{-2\pi psi}\mathcal{Y}(w_{(a)}, x)w_{(b)} \end{aligned} \tag{2.77}$$

for $w_{(a)} \in W^a$ and $w_{(b)} \in W^b$ is also an intertwining operator in $\mathcal{V}_{ab}^c$. Thus, suitably changing the intertwining operators $\mathcal{Y}_{3,i}^a$ and $\mathcal{Y}_{4,i}^a$ in this way, we see that (2.73) can be written in the form of

$$\sum_{a\in\mathcal{A}}\sum_{i=1}^{\mathcal{N}_{a_1a_2}^a\mathcal{N}_{aa_3}^{a_4}}\langle w'_{(a_4)}, \mathcal{Y}_{4,i}^a(\mathcal{Y}_{3,i}^a(w_{(a_1)},x_0)w_{(a_2)},x_2)w_{(a_3)}\rangle_{W^{a_4}}\bigg|_{x_0^n=e^{n\log(z_1-z_2)},\, x_2^n=e^{n\log z_2}}. \tag{2.78}$$

Thus on the region S_1,

$$\begin{aligned}\sum_{a\in\mathcal{A}}\sum_{i=1}^{\mathcal{N}_{a_1a_2}^a\mathcal{N}_{aa_3}^{a_4}}&\langle w'_{(a_4)}, \mathcal{Y}_{4,i}^a(\mathcal{Y}_{3,i}^a(w_{(a_1)},x_0)w_{(a_2)},x_2)w_{(a_3)}\rangle_{W^{a_4}}\bigg|_{x_0^n=e^{n\log(z_1-z_2)},\, x_2^n=e^{n\log z_2}}\\ &= \langle w'_{(a_4)}, \mathcal{Y}_1(w_{(a_1)},x_1)\mathcal{Y}_2(w_{(a_2)},x_2)w_{(a_3)}\rangle_{W^{a_4}}\big|_{x_1^n=e^{n\log z_1},\, x_2^n=e^{n\log z_2}}\end{aligned} \tag{2.79}$$

for $w_{(a_j)} \in W^{a_j}$, $j=1,2,3$, and $w'_{(a_4)} \in (W^{a_4})'$.

On the other hand, from the definition of the maps $\mathbf{P}$ and $\mathbf{I}$, we see that for any $(w_1, w_2, w_3, w') \in W^{a_{i_1}} \otimes W^{a_{i_2}} \otimes W^{a_{i_3}} \otimes (W^{a_{i_4}})'$ with $(i_1, i_2, i_3, i_4) \neq (1,2,3,4)$,

$$\begin{aligned}&\langle w', (\mathbf{P}(\mathcal{Y}_1\otimes\mathcal{Y}_2))(w_1,w_2,w_3;x_1,x_2)\rangle_W\big|_{x_1^n=e^{n\log z_1},\, x_2^n=e^{n\log z_2}}\\ &\quad= \sum_{a\in\mathcal{A}}\sum_{i=1}^{\mathcal{N}_{a_1a_2}^a\mathcal{N}_{aa_3}^{a_4}}\langle w', (\mathbf{I}(\mathcal{Y}_{3,i}^a\otimes\mathcal{Y}_{4,i}^a))(w_1,w_2,w_3;x_0,x_2)\rangle_W\bigg|_{x_0^n=e^{n\log(z_1-z_2)},\, x_2^n=e^{n\log z_2}}\\ &\quad= 0\end{aligned} \tag{2.80}$$

on S_1. So by linearity and by (2.79), (2.80), Conclusion 1 holds.

Assume that $\{\tilde{\mathcal{Y}}_{3,i}^a, \tilde{\mathcal{Y}}_{4,i}^a \mid i=1,\dots,\mathcal{N}_{a_1a_2}^a\mathcal{N}_{aa_3}^{a_4}, a\in\mathcal{A}\}$ is another set of intertwining operators satisfying Conclusion 1. Then for $w_1, w_2, w_3 \in W$ and $w' \in W'$, we have

$$\begin{aligned}\sum_{a\in\mathcal{A}}\sum_{i=1}^{\mathcal{N}_{a_1a_2}^a\mathcal{N}_{aa_3}^{a_4}}&\langle w', (\mathbf{I}(\mathcal{Y}_{3,i}^a\otimes\mathcal{Y}_{4,i}^a))(w_1,w_2,w_3;x_0,x_2)\rangle_W\bigg|_{\substack{x_0^n=e^{n\log(z_1-z_2)}\\ x_2^n=e^{n\log z_2}}}\\ &= \sum_{a\in\mathcal{A}}\sum_{i=1}^{\mathcal{N}_{a_1a_2}^a\mathcal{N}_{aa_3}^{a_4}}\langle w', (\mathbf{I}(\tilde{\mathcal{Y}}_{3,i}^a\otimes\tilde{\mathcal{Y}}_{4,i}^a))(w_1,w_2,w_3;x_0,x_2)\rangle_W\bigg|_{\substack{x_0^n=e^{n\log(z_1-z_2)}\\ x_2^n=e^{n\log z_2}}}\end{aligned} \tag{2.81}$$

on the region S_1, or equivalently,

$$\sum_{a\in\mathcal{A}}\sum_{i=1}^{\mathcal{N}_{a_1a_2}^a\mathcal{N}_{aa_3}^{a_4}}\langle w', (\mathbf{I}(\mathcal{Y}_{3,i}^a\otimes\mathcal{Y}_{4,i}^a-\tilde{\mathcal{Y}}_{3,i}^a\otimes\tilde{\mathcal{Y}}_{4,i}^a))(w_1,w_2,w_3;x_0,x_2)\rangle_W\bigg|_{\substack{x_0^n=e^{n\log(z_1-z_2)}\\ x_2^n=e^{n\log z_2}}} = 0 \tag{2.82}$$

on the region S_1. Note that the left-hand side of (2.82) is analytic in z_1 and z_2 for z_1 and z_2 satisfying $|z_2| > |z_1 - z_2| > 0$. Also note that the region S_1 is a subset of the domain $|z_2| > |z_1 - z_2| > 0$ of this analytic function. From the basic properties of analytic functions, (2.82) implies that the left-hand side of (2.82) as an analytic function is 0 for all z_1 and z_2 satisfying $|z_2| > |z_1 - z_2| > 0$. Thus for $w_1, w_2, w_3 \in W$ and $w' \in W'$,

$$\sum_{a\in\mathcal{A}} \sum_{i=1}^{\mathcal{N}_{a_1a_2}^{a}\mathcal{N}_{aa_3}^{a_4}} \langle w', (\mathbf{I}(\mathcal{Y}_{3,i}^a \otimes \mathcal{Y}_{4,i}^a - \tilde{\mathcal{Y}}_{3,i}^a \otimes \tilde{\mathcal{Y}}_{4,i}^a))(w_1, w_2, w_3; x_0, x_2)\rangle_W = 0. \tag{2.83}$$

By the definition of the map $\mathbf{I}$, (2.83) gives

$$\mathbf{I}\left(\sum_{a\in\mathcal{A}} \sum_{i=1}^{\mathcal{N}_{a_1a_2}^{a}\mathcal{N}_{aa_3}^{a_4}} (\mathcal{Y}_{3,i}^a \otimes \mathcal{Y}_{4,i}^a - \tilde{\mathcal{Y}}_{3,i}^a \otimes \tilde{\mathcal{Y}}_{4,i}^a)\right) = 0. \tag{2.84}$$

Thus

$$\sum_{a\in\mathcal{A}} \sum_{i=1}^{\mathcal{N}_{a_1a_2}^{a}\mathcal{N}_{aa_3}^{a_4}} \tilde{\mathcal{Y}}_{3,i}^a \otimes \tilde{\mathcal{Y}}_{4,i}^a \in \sum_{a\in\mathcal{A}} \sum_{i=1}^{\mathcal{N}_{a_1a_2}^{a}\mathcal{N}_{aa_3}^{a_4}} \mathcal{Y}_{3,i}^a \otimes \mathcal{Y}_{4,i}^a + \mathrm{Ker}\,\mathbf{I}, \tag{2.85}$$

proving Conclusion 2. ∎

Remark 2.10. *It is clear that (2.69) is equal to (2.70) on S_1 with $\sum_{a\in\mathcal{A}} \sum_{i=1}^{\mathcal{N}_{a_1a_2}^{a}\mathcal{N}_{aa_3}^{a_4}} \mathcal{Y}_{3,i}^a \otimes \mathcal{Y}_{4,i}^a$ replaced by any representative $\mathcal{Z}$ of $\sum_{a\in\mathcal{A}} \sum_{i=1}^{\mathcal{N}_{a_1a_2}^{a}\mathcal{N}_{aa_3}^{a_4}} \mathcal{Y}_{3,i}^a \otimes \mathcal{Y}_{4,i}^a + \mathrm{Ker}\,\mathbf{I}$. Conversely, suppose that (2.69) is equal to (2.70) on S_1 with $\sum_{a\in\mathcal{A}} \sum_{i=1}^{\mathcal{N}_{a_1a_2}^{a}\mathcal{N}_{aa_3}^{a_4}} \mathcal{Y}_{3,i}^a \otimes \mathcal{Y}_{4,i}^a$ replaced by some $\mathcal{Z} \in \coprod_{b_1,b_2,b_3,b_4,b_5\in\mathcal{A}} \mathcal{V}_{b_1b_2}^{b_5} \otimes \mathcal{V}_{b_5b_3}^{b_4}$, then in analogy with Conclusion 2 of Lemma 2.9, it can be deduced that $\mathcal{Z} \in \sum_{a\in\mathcal{A}} \sum_{i=1}^{\mathcal{N}_{a_1a_2}^{a}\mathcal{N}_{aa_3}^{a_4}} \mathcal{Y}_{3,i}^a \otimes \mathcal{Y}_{4,i}^a + \mathrm{Ker}\,\mathbf{I}$.*

From the above lemma, we can define a linear map

$$\mathcal{F}_0 : \coprod_{a_1,a_2,a_3,a_4,a_5\in\mathcal{A}} \mathcal{V}_{a_1a_5}^{a_4} \otimes \mathcal{V}_{a_2a_3}^{a_5} \longrightarrow \frac{\coprod_{a_1,a_2,a_3,a_4,a_5\in\mathcal{A}} \mathcal{V}_{a_1a_2}^{a_5} \otimes \mathcal{V}_{a_5a_3}^{a_4}}{\mathrm{Ker}\,\mathbf{I}} \tag{2.86}$$

by linearity and by

$$\mathcal{F}_0(\mathcal{Y}_1 \otimes \mathcal{Y}_2) = \sum_{a\in\mathcal{A}} \sum_{i=1}^{\mathcal{N}_{a_1a_2}^{a}\mathcal{N}_{aa_3}^{a_4}} \mathcal{Y}_{3,i}^a \otimes \mathcal{Y}_{4,i}^a + \mathrm{Ker}\,\mathbf{I} \tag{2.87}$$

for $a_1, \cdots, a_5 \in \mathcal{A}$, $\mathcal{Y}_1 \in \mathcal{V}_{a_1a_5}^{a_4}$ and $\mathcal{Y}_2 \in \mathcal{V}_{a_2a_3}^{a_5}$, where for such $\mathcal{Y}_1 \in \mathcal{V}_{a_1a_5}^{a_4}$ and $\mathcal{Y}_2 \in \mathcal{V}_{a_2a_3}^{a_5}$,

$$\{\mathcal{Y}_{3,i}^a \in \mathcal{V}_{a_1a_2}^{a}, \mathcal{Y}_{4,i}^a \in \mathcal{V}_{aa_3}^{a_4} \mid i = 1, \cdots, \mathcal{N}_{a_1a_2}^{a}\mathcal{N}_{aa_3}^{a_4}, a \in \mathcal{A}\} \tag{2.88}$$

is a set of intertwining operators satisfying Conclusion 1 of Lemma 2.9. This is indeed well defined by Conclusion 2 of Lemma 2.9. Moreover, we have:

Proposition 2.11. *The map $\mathcal{F}_0$ is surjective and* Ker $\mathcal{F}_0 =$ Ker $\mathbf{P}$*.*

Proof. Firstly, we shall show that the map $\mathcal{F}_0$ is surjective.

From Proposition 2.6 we see that, for any $a_1, \cdots, a_5 \in \mathcal{A}$, $\mathcal{Y}_3 \in \mathcal{V}_{a_1 a_2}^{a_5}$ and $\mathcal{Y}_4 \in \mathcal{V}_{a_5 a_3}^{a_4}$, there exist $\mathcal{Y}_{1,j}^{a} \in \mathcal{V}_{a_1 a}^{a_4}$ and $\mathcal{Y}_{2,j}^{a} \in \mathcal{V}_{a_2 a_3}^{a}$ for $j = 1, \ldots, \mathcal{N}_{a_1 a}^{a_4} \mathcal{N}_{a_2 a_3}^{a}$, $a \in \mathcal{A}$, such that for $w_{(a_i)} \in W^{a_i}$, $i = 1, 2, 3$, and $w'_{(a_4)} \in (W^{a_4})'$, the (multivalued) analytic function

$$\langle w'_{(a_4)}, \mathcal{Y}_4(\mathcal{Y}_3(w_{(a_1)}, x_0) w_{(a_2)}, x_2) w_{(a_3)} \rangle_{W^{a_4}} \big|_{x_0 = z_1 - z_2, x_2 = z_2} \tag{2.89}$$

defined on the region $|z_2| > |z_1 - z_2| > 0$ and the (multivalued) analytic function

$$\sum_{a \in \mathcal{A}} \sum_{j=1}^{\mathcal{N}_{a_1 a}^{a_4} \mathcal{N}_{a_2 a_3}^{a}} \langle w'_{(a_4)}, \mathcal{Y}_{1,j}^{a}(w_{(a_1)}, x_1) \mathcal{Y}_{2,j}^{a}(w_{(a_2)}, x_2) w_{(a_3)} \rangle_{W^{a_4}} \Bigg|_{x_1 = z_1, x_2 = z_2} \tag{2.90}$$

defined on the region $|z_1| > |z_2| > 0$ are equal on the intersection $|z_1| > |z_2| > |z_1 - z_2| > 0$. In analogy with the proof of Conclusion 1 of Lemma 2.9, by suitably changing the intertwining operators $\mathcal{Y}_{1,j}^{a} \in \mathcal{V}_{a_1 a}^{a_4}$ and $\mathcal{Y}_{2,j}^{a} \in \mathcal{V}_{a_2 a_3}^{a}$ for $j = 1, \ldots, \mathcal{N}_{a_1 a}^{a_4} \mathcal{N}_{a_2 a_3}^{a}$, $a \in \mathcal{A}$, we can get that

$$\left\langle w', \left(\mathbf{P} \left(\sum_{a \in \mathcal{A}} \sum_{j=1}^{\mathcal{N}_{a_1 a}^{a_4} \mathcal{N}_{a_2 a_3}^{a}} \mathcal{Y}_{1,j}^{a} \otimes \mathcal{Y}_{2,j}^{a} \right) \right) (w_1, w_2, w_3; x_1, x_2) \right\rangle_W \Bigg|_{\substack{x_1^n = e^{n \log z_1} \\ x_2^n = e^{n \log z_2}}} \tag{2.91}$$

is equal to

$$\langle w', (\mathbf{I}(\mathcal{Y}_3 \otimes \mathcal{Y}_4))(w_1, w_2, w_3; x_0, x_2) \rangle_W \big|_{x_0^n = e^{n \log(z_1 - z_2)}, x_2^n = e^{n \log z_2}} \tag{2.92}$$

on the region

$$S_1 = \{(z_1, z_2) \in \mathbb{C}^2 \mid \mathrm{Re} z_1 > \mathrm{Re} z_2 > \mathrm{Re}(z_1 - z_2) > 0,\ \mathrm{Im} z_1 > \mathrm{Im} z_2 > \mathrm{Im}(z_1 - z_2) > 0\}$$

for any $w_1, w_2, w_3 \in W$, $w' \in W'$. So by the definition of $\mathcal{F}_0$ and by Remark 2.10, we have

$$\mathcal{F}_0 \left(\sum_{a \in \mathcal{A}} \sum_{j=1}^{\mathcal{N}_{a_1 a}^{a_4} \mathcal{N}_{a_2 a_3}^{a}} \mathcal{Y}_{1,j}^{a} \otimes \mathcal{Y}_{2,j}^{a} \right) = \mathcal{Y}_3 \otimes \mathcal{Y}_4 + \mathrm{Ker}\, \mathbf{I}. \tag{2.93}$$

Therefore $\mathcal{F}_0$ is surjective by linearity.

Then we want to prove that Ker $\mathcal{F}_0 =$ Ker $\mathbf{P}$.

On the one hand, for any $\mathcal{Z} \in$ Ker $\mathcal{F}_0$, we shall prove that $\mathcal{Z} \in$ Ker $\mathbf{P}$. For any $\mathcal{Z} \in$ Ker $\mathcal{F}_0$, by Conclusion 1 of Lemma 2.9 and by linearity, there exists $\mathcal{Z}' \in \coprod_{a_1, a_2, a_3, a_4, a_5 \in \mathcal{A}} \mathcal{V}_{a_1 a_2}^{a_5} \otimes \mathcal{V}_{a_5 a_3}^{a_4}$ such that for any $w_1, w_2, w_3 \in W$ and $w' \in W'$,

$$\begin{aligned} &\langle w', (\mathbf{P}(\mathcal{Z}))(w_1, w_2, w_3; x_1, x_2) \rangle_W \big|_{x_1^n = e^{n \log z_1}, x_2^n = e^{n \log z_2}} \\ &\quad = \langle w', (\mathbf{I}(\mathcal{Z}'))(w_1, w_2, w_3; x_0, x_2) \rangle_W \big|_{x_0^n = e^{n \log(z_1 - z_2)}, x_2^n = e^{n \log z_2}} \end{aligned} \tag{2.94}$$

on the region S_1. Moreover, by the definition of $\mathcal{F}_0$ and by Remark 2.10, we have

$$\mathcal{F}_0(\mathcal{Z}) = \mathcal{Z}' + \text{Ker } \mathbf{I} = \text{Ker } \mathbf{I},$$

which implies $\mathcal{Z}' \in \text{Ker } \mathbf{I}$. So the second line of (2.94) is equal to zero on the region S_1. Note that the first line of (2.94) is analytic in z_1 and z_2 for z_1 and z_2 satisfying $|z_1| > |z_2| > 0$. Also note that S_1 is a subset of the domain $|z_1| > |z_2| > 0$ of this analytic function. From the basic properties of analytic functions, (2.94) implies that the first line of (2.94) as an analytic function is 0 for all z_1 and z_2 satisfying $|z_1| > |z_2| > 0$. Thus for any $w_1, w_2, w_3 \in W$ and $w' \in W'$,

$$\langle w', (\mathbf{P}(\mathcal{Z}))(w_1, w_2, w_3; x_1, x_2)\rangle_W = 0. \tag{2.95}$$

By the definition of $\mathbf{P}$, we therefore have $\mathbf{P}(\mathcal{Z}) = 0$; namely, $\mathcal{Z} \in \text{Ker } \mathbf{P}$.

On the other hand, we consider proving that any element $\mathcal{Z} \in \text{Ker } \mathbf{P}$ leads to $\mathcal{Z} \in \text{Ker } \mathcal{F}_0$. For any $\mathcal{Z} \in \text{Ker } \mathbf{P}$, we let $\mathcal{Z}' \in \coprod_{a_1,a_2,a_3,a_4,a_5\in\mathcal{A}} \mathcal{V}_{a_1a_2}^{a_5} \otimes \mathcal{V}_{a_5a_3}^{a_4}$ be a representative of $\mathcal{F}_0(\mathcal{Z})$. Then by the definition of $\mathcal{F}_0$ and Remark 2.10, we see that for any $w_1, w_2, w_3 \in W$ and $w' \in W'$,

$$\begin{aligned}&\langle w', (\mathbf{P}(\mathcal{Z}))(w_1, w_2, w_3; x_1, x_2)\rangle_W\big|_{x_1^n=e^{n\log z_1},\, x_2^n=e^{n\log z_2}}\\ &\quad= \langle w', (\mathbf{I}(\mathcal{Z}'))(w_1, w_2, w_3; x_0, x_2)\rangle_W\big|_{x_0^n=e^{n\log(z_1-z_2)},\, x_2^n=e^{n\log z_2}}\end{aligned} \tag{2.96}$$

on the region S_1. Moreover, (2.96) is equal to zero because $\mathcal{Z} \in \text{Ker } \mathbf{P}$. Note that the second line of (2.96) is analytic in z_1 and z_2 for z_1 and z_2 satisfying $|z_2| > |z_1 - z_2| > 0$. Also note that S_1 is a subset of the domain $|z_2| > |z_1 - z_2| > 0$ of this analytic function. From the basic properties of analytic functions, (2.96) implies that the second line of (2.96) as an analytic function is 0 for all z_1 and z_2 satisfying $|z_2| > |z_1 - z_2| > 0$. Thus for any $w_1, w_2, w_3 \in W$ and $w' \in W'$,

$$\langle w', (\mathbf{I}(\mathcal{Z}'))(w_1, w_2, w_3; x_0, x_2)\rangle_W = 0. \tag{2.97}$$

By the definition of the map $\mathbf{I}$, we therefore have $\mathbf{I}(\mathcal{Z}') = 0$; namely, $\mathcal{Z}' \in \text{Ker } \mathbf{I}$. So $\mathcal{F}_0(\mathcal{Z}) = \mathcal{Z}' + \text{Ker } \mathbf{I} = \text{Ker } \mathbf{I}$, which implies $\mathcal{Z} \in \text{Ker } \mathcal{F}_0$.

So in conclusion, we have $\text{Ker } \mathcal{F}_0 = \text{Ker } \mathbf{P}$. ∎

As a consequence of the above proposition, we have an isomorphism

$$\begin{aligned}\mathcal{F}: \quad \frac{\coprod_{a_1,a_2,a_3,a_4,a_5\in\mathcal{A}} \mathcal{V}_{a_1a_5}^{a_4} \otimes \mathcal{V}_{a_2a_3}^{a_5}}{\text{Ker } \mathbf{P}} &\longrightarrow \frac{\coprod_{a_1,a_2,a_3,a_4,a_5\in\mathcal{A}} \mathcal{V}_{a_1a_2}^{a_5} \otimes \mathcal{V}_{a_5a_3}^{a_4}}{\text{Ker } \mathbf{I}}\\ \mathcal{Z} + \text{Ker } \mathbf{P} &\longmapsto \mathcal{F}_0(\mathcal{Z}),\end{aligned} \tag{2.98}$$

where $\mathcal{Z} \in \coprod_{a_1,a_2,a_3,a_4,a_5\in\mathcal{A}} \mathcal{V}_{a_1a_5}^{a_4} \otimes \mathcal{V}_{a_2a_3}^{a_5}$. We shall call the isomorphism $\mathcal{F}$ the *fusing isomorphism*. Moreover, from $\mathcal{F}$ we can deduce an isomorphism

$$
\begin{aligned}
\mathcal{F}(a_1,a_2,a_3,a_4): \quad \pi_P\left(\coprod_{a_5\in\mathcal{A}} \mathcal{V}_{a_1a_5}^{a_4} \otimes \mathcal{V}_{a_2a_3}^{a_5}\right) &\longrightarrow \pi_I\left(\coprod_{a_5\in\mathcal{A}} \mathcal{V}_{a_1a_2}^{a_5} \otimes \mathcal{V}_{a_5a_3}^{a_4}\right)\\
\mathcal{Y}_1 \otimes \mathcal{Y}_2 + \mathrm{Ker}\,\mathbf{P} &\longmapsto \mathcal{F}(\mathcal{Y}_1 \otimes \mathcal{Y}_2 + \mathrm{Ker}\,\mathbf{P})
\end{aligned} \tag{2.99}
$$

for any $a_1, \cdots, a_4 \in \mathcal{A}$, where $\mathcal{Y}_1 \in \mathcal{V}_{a_1a_5}^{a_4}$, $\mathcal{Y}_2 \in \mathcal{V}_{a_2a_3}^{a_5}$. We also call these isomorphisms *fusing isomorphisms*.

From Ω and its inverse, we obtain the following linear maps:

$$
\tilde{\Omega}^{(1)}, (\widetilde{\Omega^{-1}})^{(1)}: \ \frac{\coprod_{a_1,a_2,a_3,a_4,a_5\in\mathcal{A}} \mathcal{V}_{a_1a_2}^{a_5} \otimes \mathcal{V}_{a_5a_3}^{a_4}}{\mathrm{Ker}\,\mathbf{I}} \longrightarrow \frac{\coprod_{a_1,a_2,a_3,a_4,a_5\in\mathcal{A}} \mathcal{V}_{a_2a_1}^{a_5} \otimes \mathcal{V}_{a_5a_3}^{a_4}}{\mathrm{Ker}\,\mathbf{I}} \tag{2.100}
$$

determined by linearity and by

$$
\tilde{\Omega}^{(1)}(\mathcal{Y}_1 \otimes \mathcal{Y}_2 + \mathrm{Ker}\,\mathbf{I}) = \Omega(\mathcal{Y}_1) \otimes \mathcal{Y}_2 + \mathrm{Ker}\,\mathbf{I}, \tag{2.101}
$$

$$
(\widetilde{\Omega^{-1}})^{(1)}(\mathcal{Y}_1 \otimes \mathcal{Y}_2 + \mathrm{Ker}\,\mathbf{I}) = \Omega^{-1}(\mathcal{Y}_1) \otimes \mathcal{Y}_2 + \mathrm{Ker}\,\mathbf{I} \tag{2.102}
$$

for $a_1, \dots, a_5 \in \mathcal{A}$, $\mathcal{Y}_1 \in \mathcal{V}_{a_1a_2}^{a_5}$ and $\mathcal{Y}_2 \in \mathcal{V}_{a_5a_3}^{a_4}$.

Proposition 2.12. *The two linear maps $\tilde{\Omega}^{(1)}$ and $(\widetilde{\Omega^{-1}})^{(1)}$ are well defined and are isomorphisms. Moreover, they are inverse to each other.*

Proof. For any

$$
\sum_{a_1,\cdots,a_5\in\mathcal{A}} \sum_{i=1}^{m} \mathcal{Y}_{a_1a_2,i}^{a_5} \otimes \mathcal{Y}_{a_5a_3,i}^{a_4} \in \mathrm{Ker}\,\mathbf{I}, \tag{2.103}
$$

where $\mathcal{Y}_{a_1a_2,i}^{a_5} \in \mathcal{V}_{a_1a_2}^{a_5}$, $\mathcal{Y}_{a_5a_3,i}^{a_4} \in \mathcal{V}_{a_5a_3}^{a_4}$ for $a_1, \cdots, a_5 \in \mathcal{A}$ and m is some non-negative integer, we have

$$
\begin{aligned}
&\left(\mathbf{I}\left(\sum_{a_1,\cdots,a_5\in\mathcal{A}} \sum_{i=1}^{m} \Omega(\mathcal{Y}_{a_1a_2,i}^{a_5}) \otimes \mathcal{Y}_{a_5a_3,i}^{a_4}\right)\right)(w_1, w_2, w_3; x_0, x_2)\\
&\quad = \left(\mathbf{I}\left(\sum_{a_1,\cdots,a_5\in\mathcal{A}} \sum_{i=1}^{m} \mathcal{Y}_{a_1a_2,i}^{a_5} \otimes \mathcal{Y}_{a_5a_3,i}^{a_4}\right)\right)(w_2, w_1, w_3; e^{-\pi i}x_0, x_1)\\
&\quad = 0
\end{aligned} \tag{2.104}
$$

for any $w_1, w_2, w_3 \in W$. So we have

$$
\tilde{\Omega}^{(1)}\left(\sum_{a_1,\cdots,a_5\in\mathcal{A}} \sum_{i=1}^{m} \mathcal{Y}_{a_1a_2,i}^{a_5} \otimes \mathcal{Y}_{a_5a_3,i}^{a_4}\right) = \sum_{a_1,\cdots,a_5\in\mathcal{A}} \sum_{i=1}^{m} \Omega(\mathcal{Y}_{a_1a_2,i}^{a_5}) \otimes \mathcal{Y}_{a_5a_3,i}^{a_4} \in \mathrm{Ker}\,\mathbf{I}. \tag{2.105}
$$

Thus $\tilde{\Omega}^{(1)}$ is well defined. Similarly, we can prove that $(\widetilde{\Omega^{-1}})^{(1)}$ is well defined.

By definition, $\tilde{\Omega}^{(1)}$ and $(\widetilde{\Omega^{-1}})^{(1)}$ are inverse to each other. Thus they are both isomorphisms. ∎

Using the fusing isomorphism and the skew-symmetry isomorphism, we define a *braiding isomorphism*

$$\mathcal{B} = \mathcal{F}^{-1} \circ \tilde{\Omega}^{(1)} \circ \mathcal{F} : \frac{\coprod_{a_1,a_2,a_3,a_4,a_5 \in \mathcal{A}} \mathcal{V}_{a_1 a_5}^{a_4} \otimes \mathcal{V}_{a_2 a_3}^{a_5}}{\text{Ker } \mathbf{P}} \longrightarrow \frac{\coprod_{a_1,a_2,a_3,a_4,a_5 \in \mathcal{A}} \mathcal{V}_{a_2 a_5}^{a_4} \otimes \mathcal{V}_{a_1 a_3}^{a_5}}{\text{Ker } \mathbf{P}}. \tag{2.106}$$

Moreover, we have an isomorphism

$$\begin{aligned} \mathcal{B}(a_1, a_2, a_3, a_4) : \quad \pi_P \left(\coprod_{a_5 \in \mathcal{A}} \mathcal{V}_{a_1 a_5}^{a_4} \otimes \mathcal{V}_{a_2 a_3}^{a_5} \right) &\longrightarrow \pi_P \left(\coprod_{a_5 \in \mathcal{A}} \mathcal{V}_{a_2 a_5}^{a_4} \otimes \mathcal{V}_{a_1 a_3}^{a_5} \right) \\ \mathcal{Y}_1 \otimes \mathcal{Y}_2 + \text{Ker } \mathbf{P} &\longmapsto \mathcal{B}(\mathcal{Y}_1 \otimes \mathcal{Y}_2 + \text{Ker } \mathbf{P}) \end{aligned} \tag{2.107}$$

for any $a_1, \cdots, a_4 \in \mathcal{A}$, where $\mathcal{Y}_1 \in \mathcal{V}_{a_1 a_5}^{a_4}$, $\mathcal{Y}_2 \in \mathcal{V}_{a_2 a_3}^{a_5}$. We also call these isomorphisms *braiding isomorphisms*.

We now reformulate the associativity and commutativity properties for intertwining operators using the fusing and braiding isomorphisms as follows:

Associativity: For any $a_1, a_2, a_3, a_4, a_5 \in \mathcal{A}$, $\mathcal{Y}_1 \in \mathcal{V}_{a_1 a_5}^{a_4}$, $\mathcal{Y}_2 \in \mathcal{V}_{a_2 a_3}^{a_5}$, $w_1, w_2, w_3 \in W$ and $w' \in W'$, the (multivalued) analytic function

$$\langle w', (\tilde{\mathbf{P}}([\mathcal{Y}_1 \otimes \mathcal{Y}_2]_P))(w_1, w_2, w_3; x_1, x_2)\rangle_W|_{x_1=z_1, x_2=z_2} \tag{2.108}$$

defined on the region $|z_1| > |z_2| > 0$ and the (multivalued) analytic function

$$\langle w', (\tilde{\mathbf{I}}(\mathcal{F}([\mathcal{Y}_1 \otimes \mathcal{Y}_2]_P)))(w_1, w_2, w_3; x_0, x_2)\rangle_W|_{x_0=z_1-z_2, x_2=z_2} \tag{2.109}$$

defined on the region $|z_2| > |z_1 - z_2| > 0$ are equal on the intersection $|z_1| > |z_2| > |z_1 - z_2| > 0$. In addition,

$$\begin{aligned} &\langle w', (\tilde{\mathbf{P}}([\mathcal{Y}_1 \otimes \mathcal{Y}_2]_P))(w_1, w_2, w_3; x_1, x_2)\rangle_W \Big|_{\substack{x_1^n = e^{n \log z_1} \\ x_2^n = e^{n \log z_2}}} \\ &\quad = \langle w', (\tilde{\mathbf{I}}(\mathcal{F}([\mathcal{Y}_1 \otimes \mathcal{Y}_2]_P)))(w_1, w_2, w_3; x_0, x_2)\rangle_W \Big|_{\substack{x_0^n = e^{n \log(z_1 - z_2)} \\ x_2^n = e^{n \log z_2}}} \end{aligned} \tag{2.110}$$

on the simply connected region

$$S_1 = \{(z_1, z_2) \in \mathbb{C}^2 \mid \text{Re} z_1 > \text{Re} z_2 > \text{Re}(z_1 - z_2) > 0, \ \text{Im} z_1 > \text{Im} z_2 > \text{Im}(z_1 - z_2) > 0\}.$$

Commutativity: For any $a_1, a_2, a_3, a_4, a_5 \in \mathcal{A}$, $\mathcal{Y}_1 \in \mathcal{V}_{a_1 a_5}^{a_4}$, $\mathcal{Y}_2 \in \mathcal{V}_{a_2 a_3}^{a_5}$, $w_1, w_2, w_3 \in W$ and $w' \in W'$, the (multivalued) analytic function (2.108) on the region $|z_1| > |z_2| > 0$ and the (multivalued) analytic function

$$\langle w', (\tilde{\mathbf{P}}(\mathcal{B}([\mathcal{Y}_1 \otimes \mathcal{Y}_2]_P)))(w_2, w_1, w_3; x_2, x_1)\rangle_W|_{x_1=z_1, x_2=z_2} \tag{2.111}$$

on the region $|z_2| > |z_1| > 0$ are analytic extensions of each other.

For simplicity, as we have mentioned before, we have used $[\mathcal{Y}_1 \otimes \mathcal{Y}_2]_P$ above to denote $\mathcal{Y}_1 \otimes \mathcal{Y}_2 + \mathrm{Ker}\,\mathbf{P}$.

Remark 2.13. *From the associativity, commutativity and convergence properties of intertwining operator algebras, we can deduce that for any $a_1, a_2, a_3, a_4, a_5 \in \mathcal{A}$, $\mathcal{Y}_1 \in \mathcal{V}_{a_1 a_5}^{a_4}$ and $\mathcal{Y}_2 \in \mathcal{V}_{a_2 a_3}^{a_5}$, there exists a multivalued analytic function defined on $M^2 = \{(z_1, z_2) \in \mathbb{C}^2 \mid z_1, z_2 \neq 0, z_1 \neq z_2\}$ such that (2.108), (2.109) and (2.111) are parts of the restrictions of this function to the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1 - z_2| > 0$ and $|z_2| > |z_1| > 0$, respectively.*

Remark 2.14. *The commutativity property (2.111) and Remark 2.13 also hold with the fusing isomorphism $\mathcal{B}$ replaced by its inverse $\mathcal{B}^{-1}$.*

Moreover, we have the following generalized rationality of products and commutativity in formal variables formulated and proved in [H9]:

Theorem 2.15. *For $a_1, a_2, a_3, a_4 \in \mathcal{A}$, there exist linear maps*

$$\begin{aligned} f_\alpha^{a_1,a_2,a_3,a_4} : W^{a_1} \otimes W^{a_2} \otimes W^{a_3} \otimes \pi_P &\left(\coprod_{a_5 \in \mathcal{A}} \mathcal{V}_{a_1 a_5}^{a_4} \otimes \mathcal{V}_{a_2 a_3}^{a_5}\right) \\ &\to W^{a_4}[[x_2/x_1]][x_1, x_1^{-1}, x_2, x_2^{-1}] \\ w_{(a_1)} \otimes w_{(a_2)} \otimes w_{(a_3)} \otimes [\mathcal{Z}]_P & \\ &\mapsto f_\alpha^{a_1,a_2,a_3,a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, [\mathcal{Z}]_P; x_1, x_2) \end{aligned} \tag{2.112}$$

and

$$\begin{aligned} g_\alpha^{a_1,a_2,a_3,a_4} : W^{a_1} \otimes W^{a_2} \otimes W^{a_3} \otimes \pi_P &\left(\coprod_{a_5 \in \mathcal{A}} \mathcal{V}_{a_2 a_5}^{a_4} \otimes \mathcal{V}_{a_1 a_3}^{a_5}\right) \\ &\to W^{a_4}[[x_1/x_2]][x_1, x_1^{-1}, x_2, x_2^{-1}] \\ w_{(a_1)} \otimes w_{(a_2)} \otimes w_{(a_3)} \otimes [\mathcal{Z}]_P & \\ &\mapsto g_\alpha^{a_1,a_2,a_3,a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, [\mathcal{Z}]_P; x_1, x_2), \end{aligned} \tag{2.113}$$

$\alpha \in \mathbb{A}(a_1, a_2, a_3, a_4)$*, satisfying the following generalized rationality of products and commutativity: For any* $w_{(a_1)} \in W^{a_1}$*,* $w_{(a_2)} \in W^{a_2}$*,* $w_{(a_3)} \in W^{a_3}$*,* $w'_{(a_4)} \in (W^{a_4})'$*, and any*

$$\mathcal{Z} \in \coprod_{a_5 \in \mathcal{A}} \mathcal{V}_{a_1 a_5}^{a_4} \otimes \mathcal{V}_{a_2 a_3}^{a_5}, \tag{2.114}$$

only finitely many of

$$f_\alpha^{a_1,a_2,a_3,a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, [\mathcal{Z}]_P; x_1, x_2), \tag{2.115}$$

$$g_\alpha^{a_1,a_2,a_3,a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, \mathcal{B}([\mathcal{Z}]_P); x_1, x_2), \tag{2.116}$$

$\alpha \in \mathbb{A}(a_1, a_2, a_3, a_4)$*, are nonzero, and there exist*

$$F_\alpha(w'_{(a_4)}, w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, [\mathcal{Z}]_P; x_1, x_2) \in \mathbb{C}[x_1, x_1^{-1}, x_2, x_2^{-1}, (x_1 - x_2)^{-1}] \tag{2.117}$$

for $\alpha \in \mathbb{A}(a_1, a_2, a_3, a_4)$*, such that*

$$\begin{aligned}(\tilde{\mathbf{P}}([\mathcal{Z}]_P))&(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}; x_1, x_2)\\ &= \sum_{\alpha \in \mathbb{A}(a_1,a_2,a_3,a_4)} f_\alpha^{a_1,a_2,a_3,a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, [\mathcal{Z}]_P; x_1, x_2)\iota_{12}(e_\alpha^{a_1,a_2,a_3,a_4}),\end{aligned} \tag{2.118}$$

$$\begin{aligned}(\tilde{\mathbf{P}}(\mathcal{B}([\mathcal{Z}]_P)))&(w_{(a_2)}, w_{(a_1)}, w_{(a_3)}; x_2, x_1)\\ &= \sum_{\alpha \in \mathbb{A}(a_1,a_2,a_3,a_4)} g_\alpha^{a_1,a_2,a_3,a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, \mathcal{B}([\mathcal{Z}]_P); x_1, x_2)\iota_{21}(e_\alpha^{a_1,a_2,a_3,a_4}),\end{aligned} \tag{2.119}$$

and

$$\begin{aligned}\langle w'_{(a_4)}, &f_\alpha^{a_1,a_2,a_3,a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, [\mathcal{Z}]_P; x_1, x_2)\rangle_{W^{a_4}}\\ &= \iota_{12}(F_\alpha(w'_{(a_4)}, w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, [\mathcal{Z}]_P; x_1, x_2)),\end{aligned} \tag{2.120}$$

$$\begin{aligned}\langle w'_{(a_4)}, &g_\alpha^{a_1,a_2,a_3,a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, \mathcal{B}([\mathcal{Z}]_P); x_1, x_2)\rangle_{W^{a_4}}\\ &= \iota_{21}(F_\alpha(w'_{(a_4)}, w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, [\mathcal{Z}]_P; x_1, x_2))\end{aligned} \tag{2.121}$$

for $\alpha \in \mathbb{A}(a_1, a_2, a_3, a_4)$*.*

It is clear that commutativity (Proposition 2.7, eq. (2.111)) follows from generalized rationality of products and commutativity in formal variables.

We also have the following generalized rationality of iterates and associativity in formal variables formulated and proved in [H9]:

Theorem 2.16. *For $a_1, a_2, a_3, a_4 \in \mathcal{A}$, there exist linear maps*

$$\begin{aligned}
h_\alpha^{a_1,a_2,a_3,a_4} : W^{a_1} \otimes W^{a_2} \otimes W^{a_3} \otimes \pi_I(\coprod_{a_5 \in \mathcal{A}} \mathcal{V}_{a_1a_2}^{a_5} \otimes \mathcal{V}_{a_5a_3}^{a_4}) \\
\to W^{a_4}[[x_0/x_2]][x_0, x_0^{-1}, x_2, x_2^{-1}] \\
w_{(a_1)} \otimes w_{(a_2)} \otimes w_{(a_3)} \otimes [\mathcal{Z}]_I \\
\mapsto h_\alpha^{a_1,a_2,a_3,a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, [\mathcal{Z}]_I; x_0, x_2),
\end{aligned} \tag{2.122}$$

$\alpha \in \mathbb{A}(a_1, a_2, a_3, a_4)$, satisfying the following generalized rationality of iterates and associativity: For any $w_{(a_1)} \in W^{a_1}$, $w_{(a_2)} \in W^{a_2}$, $w_{(a_3)} \in W^{a_3}$, $w'_{(a_4)} \in (W^{a_4})'$, and any

$$\mathcal{Z} \in \coprod_{a_5 \in \mathcal{A}} \mathcal{V}_{a_1a_5}^{a_4} \otimes \mathcal{V}_{a_2a_3}^{a_5}, \tag{2.123}$$

only finitely many of

$$h_\alpha^{a_1,a_2,a_3,a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, \mathcal{F}([\mathcal{Z}]_P); x_0, x_2), \tag{2.124}$$

$\alpha \in \mathbb{A}(a_1, a_2, a_3, a_4)$, are nonzero, and we have

$$\begin{aligned}
&(\tilde{\mathbf{I}}(\mathcal{F}([\mathcal{Z}]_P)))(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}; x_0, x_2) \\
&\quad = \sum_{\alpha \in \mathbb{A}(a_1,a_2,a_3,a_4)} h_\alpha^{a_1,a_2,a_3,a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, \mathcal{F}([\mathcal{Z}]_P); x_0, x_2)\iota_{20}(e_\alpha^{a_1,a_2,a_3,a_4})
\end{aligned} \tag{2.125}$$

and

$$\begin{aligned}
&\langle w'_{(a_4)}, h_\alpha^{a_1,a_2,a_3,a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, \mathcal{F}([\mathcal{Z}]_P); x_0, x_2)\rangle_{W^{a_4}} \\
&\quad = \iota_{20}(F_\alpha(w'_{(a_4)}, w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, [\mathcal{Z}]_P; x_2 + x_0, x_2))
\end{aligned} \tag{2.126}$$

for $\alpha \in \mathbb{A}(a_1, a_2, a_3, a_4)$.

It is clear that associativity (Axiom 4 of Definition 2.3, eq. (2.108)-(2.110)) follows from generalized rationality of iterates and associativity in formal variables.

From the generalized rationality, commutativity and associativity in formal variables, the following Jacobi identity was derived in [H9]:

Theorem 2.17 (Jacobi identity). *For any $a_1, a_2, a_3, a_4 \in \mathcal{A}$, there exist linear maps*

$$\begin{aligned}
f_\alpha^{a_1,a_2,a_3,a_4} : W^{a_1} \otimes W^{a_2} \otimes W^{a_3} \otimes \pi_P(\coprod_{a_5 \in \mathcal{A}} \mathcal{V}_{a_1a_5}^{a_4} \otimes \mathcal{V}_{a_2a_3}^{a_5}) \\
\to W^{a_4}[[x_2/x_1]][x_1, x_1^{-1}, x_2, x_2^{-1}] \\
w_{(a_1)} \otimes w_{(a_2)} \otimes w_{(a_3)} \otimes [\mathcal{Z}]_P \\
\mapsto f_\alpha^{a_1,a_2,a_3,a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, [\mathcal{Z}]_P; x_1, x_2),
\end{aligned} \tag{2.127}$$

$$\begin{aligned} g_\alpha^{a_1,a_2,a_3,a_4} : W^{a_1} \otimes W^{a_2} \otimes W^{a_3} \otimes \pi_P(\coprod_{a_5\in\mathcal{A}} \mathcal{V}_{a_2a_5}^{a_4} \otimes \mathcal{V}_{a_1a_3}^{a_5}) \\ \to W^{a_4}[[x_1/x_2]][x_1, x_1^{-1}, x_2, x_2^{-1}] \\ w_{(a_1)} \otimes w_{(a_2)} \otimes w_{(a_3)} \otimes [\mathcal{Z}]_P \\ \mapsto g_\alpha^{a_1,a_2,a_3,a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, [\mathcal{Z}]_P; x_1, x_2) \end{aligned} \tag{2.128}$$

and

$$\begin{aligned} h_\alpha^{a_1,a_2,a_3,a_4} : W^{a_1} \otimes W^{a_2} \otimes W^{a_3} \otimes \pi_I(\coprod_{a_5\in\mathcal{A}} \mathcal{V}_{a_1a_2}^{a_5} \otimes \mathcal{V}_{a_5a_3}^{a_4}) \\ \to W^{a_4}[[x_0/x_2]][x_0, x_0^{-1}, x_2, x_2^{-1}] \\ w_{(a_1)} \otimes w_{(a_2)} \otimes w_{(a_3)} \otimes [\mathcal{Z}]_I \\ \mapsto h_\alpha^{a_1,a_2,a_3,a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, [\mathcal{Z}]_I; x_0, x_2) \end{aligned} \tag{2.129}$$

for $\alpha \in \mathbb{A}(a_1, a_2, a_3, a_4)$*, such that for any* $w_{(a_1)} \in W^{a_1}$*,* $w_{(a_2)} \in W^{a_2}$*,* $w_{(a_3)} \in W^{a_3}$*, and any*

$$\mathcal{Z} \in \coprod_{a_5\in\mathcal{A}} \mathcal{V}_{a_1a_5}^{a_4} \otimes \mathcal{V}_{a_2a_3}^{a_5} \subset \coprod_{a_1,a_2,a_3,a_4,a_5\in\mathcal{A}} \mathcal{V}_{a_1a_5}^{a_4} \otimes \mathcal{V}_{a_2a_3}^{a_5}, \tag{2.130}$$

only finitely many of

$$f_\alpha^{a_1,a_2,a_3,a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, [\mathcal{Z}]_P; x_1, x_2), \tag{2.131}$$

$$g_\alpha^{a_1,a_2,a_3,a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, \mathcal{B}([\mathcal{Z}]_P); x_1, x_2), \tag{2.132}$$

and

$$h_\alpha^{a_1,a_2,a_3,a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, \mathcal{F}([\mathcal{Z}]_P); x_0, x_2), \tag{2.133}$$

$\alpha \in \mathbb{A}(a_1, a_2, a_3, a_4)$*, are nonzero,*

$$\begin{aligned} &(\tilde{\mathbf{P}}([\mathcal{Z}]_P))(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}; x_1, x_2) \\ &\quad = \sum_{\alpha\in\mathbb{A}(a_1,a_2,a_3,a_4)} f_\alpha^{a_1,a_2,a_3,a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, [\mathcal{Z}]_P; x_1, x_2)\iota_{12}\left(e_\alpha^{a_1,a_2,a_3,a_4}\right), \end{aligned} \tag{2.134}$$

$$\begin{aligned} &(\tilde{\mathbf{P}}(\mathcal{B}([\mathcal{Z}]_P)))(w_{(a_2)}, w_{(a_1)}, w_{(a_3)}; x_2, x_1) \\ &\quad = \sum_{\alpha\in\mathbb{A}(a_1,a_2,a_3,a_4)} g_\alpha^{a_1,a_2,a_3,a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, \mathcal{B}([\mathcal{Z}]_P); x_1, x_2)\iota_{21}\left(e_\alpha^{a_1,a_2,a_3,a_4}\right), \end{aligned} \tag{2.135}$$

$$\begin{aligned} &(\tilde{\mathbf{I}}(\mathcal{F}([\mathcal{Z}]_P)))(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}; x_0, x_2) \\ &\quad = \sum_{\alpha\in\mathbb{A}(a_1,a_2,a_3,a_4)} h_\alpha^{a_1,a_2,a_3,a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, \mathcal{F}([\mathcal{Z}]_P); x_0, x_2)\iota_{20}(e_\alpha^{a_1,a_2,a_3,a_4}), \end{aligned} \tag{2.136}$$

and the following Jacobi identity holds:

$$\begin{aligned}
&x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right) f_\alpha^{a_1,a_2,a_3,a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, [\mathcal{Z}]_P; x_1, x_2)\\
&\quad -x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right) g_\alpha^{a_1,a_2,a_3,a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, \mathcal{B}([\mathcal{Z}]_P); x_1, x_2)\\
&\quad = x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right) h_\alpha^{a_1,a_2,a_3,a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, \mathcal{F}([\mathcal{Z}]_P); x_0, x_2)
\end{aligned} \tag{2.137}$$

for $\alpha \in \mathbb{A}(a_1, a_2, a_3, a_4)$.

3 Duality properties and the Jacobi identity

In this section, we study the relations among commutativity, associativity, skew-symmetry and Jacobi identity. We introduce and prove locality for intertwining operator algebras.

It has been proved in the preceding section that associativity and skew-symmetry imply the second associativity and commutativity. We now show that commutativity and skew-symmetry imply associativity.

Theorem 3.1. *In the presence of the axioms for intertwining operator algebras except for associativity and skew-symmetry, the associativity property of intertwining operator algebras follows from commutativity and skew-symmetry.*

Proof. By the skew-symmetry property, on the region $\{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1| > |z_2| > 0, |z_1 - z_2| > |z_2| > 0\}$, we have

$$\begin{aligned}
&\langle w'_{(a_4)}, \mathcal{Y}_1(w_{(a_1)}, x_1)\mathcal{Y}_2(w_{(a_2)}, x_2)w_{(a_3)}\rangle_{W^{a_4}}\big|_{x_1=z_1, x_2=z_2}\\
&\quad = \langle w'_{(a_4)}, \mathcal{Y}_1(w_{(a_1)}, x_1)\Omega^{-1}(\Omega(\mathcal{Y}_2))(w_{(a_2)}, x_2)w_{(a_3)}\rangle_{W^{a_4}}\big|_{x_1=z_1, x_2=z_2}\\
&\quad = \langle w'_{(a_4)}, \mathcal{Y}_1(w_{(a_1)}, x_1)e^{x_2L(-1)}\Omega(\mathcal{Y}_2)(w_{(a_3)}, e^{\pi i}x_2)w_{(a_2)}\rangle_{W^{a_4}}\big|_{x_1=z_1, x_2=z_2}\\
&\quad = \langle e^{x_2L(1)}w'_{(a_4)}, \mathcal{Y}_1(w_{(a_1)}, x_0)\Omega(\mathcal{Y}_2)(w_{(a_3)}, e^{\pi i}x_2)w_{(a_2)}\rangle_{W^{a_4}}\big|_{x_0=z_1-z_2, x_2=z_2}
\end{aligned} \tag{3.1}$$

for any $\mathcal{Y}_1 \in \mathcal{V}_{a_1a_5}^{a_4}$, $\mathcal{Y}_2 \in \mathcal{V}_{a_2a_3}^{a_5}$, $w_{(a_i)} \in W^{a_i}$, $i = 1, 2, 3$, and $w'_{(a_4)} \in (W^{a_4})'$. By the commutativity property, there exist $\mathcal{Y}_{5,i}^a \in \mathcal{V}_{a_1a_2}^a$ and $\mathcal{Y}_{6,i}^a \in \mathcal{V}_{a_3a}^{a_4}$ for $i = 1, \dots, \mathcal{N}_{a_1a_2}^a\mathcal{N}_{a_3a}^{a_4}$ and $a \in \mathcal{A}$, such that the (multivalued) analytic function given by the last line of (3.1) defined on the region $|z_1 - z_2| > |z_2| > 0$ and the (multivalued) analytic function

$$\sum_{a\in\mathcal{A}}\sum_{i=1}^{\mathcal{N}_{a_1a_2}^a\mathcal{N}_{a_3a}^{a_4}} \langle e^{x_2L(1)}w'_{(a_4)}, \mathcal{Y}_{6,i}^a(w_{(a_3)}, e^{\pi i}x_2)\mathcal{Y}_{5,i}^a(w_{(a_1)}, x_0)w_{(a_2)}\rangle_{W^{a_4}}\Bigg|_{x_0=z_1-z_2, x_2=z_2} \tag{3.2}$$

defined on the region $|z_2| > |z_1 - z_2| > 0$ are analytic extensions of each other. Moreover, by skew-symmetry again, on the region $|z_2| > |z_1 - z_2| > 0$, we have

$$
\begin{aligned}
&\sum_{a\in\mathcal{A}} \sum_{i=1}^{\mathcal{N}_{a_1a_2}^{a}\mathcal{N}_{a_3a}^{a_4}} \langle e^{x_2L(1)}w'_{(a_4)}, \mathcal{Y}_{6,i}^{a}(w_{(a_3)}, e^{\pi i}x_2)\mathcal{Y}_{5,i}^{a}(w_{(a_1)}, x_0)w_{(a_2)}\rangle_{W^{a_4}}\Big|_{\substack{x_0=z_1-z_2\\ x_2=z_2}}\\
&\quad = \sum_{a\in\mathcal{A}} \sum_{i=1}^{\mathcal{N}_{a_1a_2}^{a}\mathcal{N}_{a_3a}^{a_4}} \langle e^{x_2L(1)}w'_{(a_4)}, \Omega(\Omega^{-1}(\mathcal{Y}_{6,i}^{a}))(w_{(a_3)}, e^{\pi i}x_2)\mathcal{Y}_{5,i}^{a}(w_{(a_1)}, x_0)w_{(a_2)}\rangle_{W^{a_4}}\Big|_{\substack{x_0=z_1-z_2\\ x_2=z_2}}\\
&\quad = \sum_{a\in\mathcal{A}} \sum_{i=1}^{\mathcal{N}_{a_1a_2}^{a}\mathcal{N}_{a_3a}^{a_4}} \langle e^{x_2L(1)}w'_{(a_4)}, e^{-x_2L(-1)}\Omega^{-1}(\mathcal{Y}_{6,i}^{a})(\mathcal{Y}_{5,i}^{a}(w_{(a_1)}, x_0)w_{(a_2)}, x_2)w_{(a_3)}\rangle_{W^{a_4}}\Big|_{\substack{x_0=z_1-z_2\\ x_2=z_2}}\\
&\quad = \sum_{a\in\mathcal{A}} \sum_{i=1}^{\mathcal{N}_{a_1a_2}^{a}\mathcal{N}_{aa_3}^{a_4}} \langle w'_{(a_4)}, \Omega^{-1}(\mathcal{Y}_{6,i}^{a})(\mathcal{Y}_{5,i}^{a}(w_{(a_1)}, x_0)w_{(a_2)}, x_2)w_{(a_3)}\rangle_{W^{a_4}}\Big|_{\substack{x_0=z_1-z_2\\ x_2=z_2}},
\end{aligned}
\tag{3.3}
$$

where $\mathcal{N}_{aa_3}^{a_4} = \mathcal{N}_{a_3a}^{a_4}$. Taking $\mathcal{Y}_{3,i}^{a} = \mathcal{Y}_{5,i}^{a}$ and $\mathcal{Y}_{4,i}^{a} = \Omega^{-1}(\mathcal{Y}_{6,i}^{a})$ for $i = 1, \dots, \mathcal{N}_{a_1a_2}^{a}\mathcal{N}_{aa_3}^{a_4}$ and $a \in \mathcal{A}$, we see that the (multivalued) analytic function

$$\langle w'_{(a_4)}, \mathcal{Y}_1(w_{(a_1)}, x_1)\mathcal{Y}_2(w_{(a_2)}, x_2)w_{(a_3)}\rangle_{W^{a_4}}|_{x_1=z_1,x_2=z_2} \tag{3.4}$$

defined on the region $|z_1| > |z_2| > 0$ and the (multivalued) analytic function

$$\sum_{a\in\mathcal{A}} \sum_{i=1}^{\mathcal{N}_{a_1a_2}^{a}\mathcal{N}_{aa_3}^{a_4}} \langle w'_{(a_4)}, \mathcal{Y}_{4,i}^{a}(\mathcal{Y}_{3,i}^{a}(w_{(a_1)}, x_0)w_{(a_2)}, x_2)w_{(a_3)}\rangle_{W^{a_4}}|_{x_0=z_1-z_2,x_2=z_2} \tag{3.5}$$

defined on the region $|z_2| > |z_1 - z_2| > 0$ are equal on the intersection $|z_1| > |z_2| > |z_1 - z_2| > 0$. This proves the associativity property. ∎

In the theory of vertex operator algebras, commutativity and associativity imply Jacobi identity and Jacobi identity implies skew-symmetry [FHL]. Thus commutativity and associativity imply skew-symmetry. In the theory of intertwining operator algebras, we have the following generalization:

Theorem 3.2. *In the presence of the axioms for intertwining operator algebras except for skew-symmetry, we assume that commutativity holds, and that the restriction of Ω to $\mathcal{V}_{ea}^{b}$ is an isomorphism from $\mathcal{V}_{ea}^{b}$ to $\mathcal{V}_{ae}^{b}$ for any $a, b \in \mathcal{A}$. Then the restriction of Ω to $\mathcal{V}_{a_1a_2}^{a_3}$ is an isomorphism from $\mathcal{V}_{a_1a_2}^{a_3}$ to $\mathcal{V}_{a_2a_1}^{a_3}$ for any $a_1, a_2, a_3 \in \mathcal{A}$.*

Proof. Recall that the vector space $\mathcal{V}_{ea}^{a}$ for any $a \in \mathcal{A}$ is the one-dimensional vector space spanned by the vertex operator for the W^e-module W^a, and that $\mathcal{V}_{ea}^{b} = 0$ for any

$a, b \in \mathcal{A}$ with $a \neq b$. So by the assumption that the restriction of Ω to $\mathcal{V}_{ea}^{b}$ is an isomorphism from $\mathcal{V}_{ea}^{b}$ to $\mathcal{V}_{ae}^{b}$ for any $a, b \in \mathcal{A}$, we have

$$\dim \mathcal{V}_{ae}^{a} = 1 \text{ and } \mathcal{V}_{ae}^{b} = 0 \text{ for any } a, b \in \mathcal{A} \text{ with } a \neq b. \tag{3.6}$$

For any $a_1, a_2, a_3 \in \mathcal{A}$, $\mathcal{Y} \in \mathcal{V}_{a_1 a_2}^{a_3}$, $w_{(a_1)} \in W^{a_1}$, $w_{(a_2)} \in W^{a_2}$ and $w'_{(a_3)} \in (W^{a_3})'$, on the region $\{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1| > |z_2| > 0, |z_1 - z_2| > |z_2| > 0\}$, we have

$$\begin{aligned}
&\langle w'_{(a_3)}, \Omega(\mathcal{Y})(w_{(a_2)}, x_1) w_{(a_1)} \rangle_{W^{a_3}} \big|_{x_1 = z_1} \\
&\quad = \langle w'_{(a_3)}, e^{x_1 L(-1)} \mathcal{Y}(w_{(a_1)}, e^{-\pi i} x_1) w_{(a_2)} \rangle_{W^{a_3}} \big|_{x_1 = z_1} \\
&\quad = \langle w'_{(a_3)}, e^{x_1 L(-1)} \mathcal{Y}(w_{(a_1)}, e^{-\pi i} x_1) Y_{a_2}(\mathbf{1}, e^{-\pi i} x_2) w_{(a_2)} \rangle_{W^{a_3}} \big|_{x_1 = z_1, x_2 = z_2} \\
&\quad = \langle w'_{(a_3)}, e^{x_1 L(-1)} \mathcal{Y}(w_{(a_1)}, e^{-\pi i} x_1) e^{-x_2 L(-1)} \Omega(Y_{a_2})(w_{(a_2)}, x_2) \mathbf{1} \rangle_{W^{a_3}} \big|_{x_1 = z_1, x_2 = z_2} \\
&\quad = \langle w'_{(a_3)}, e^{x_0 L(-1)} \mathcal{Y}(w_{(a_1)}, e^{-\pi i} x_0) \Omega(Y_{a_2})(w_{(a_2)}, x_2) \mathbf{1} \rangle_{W^{a_3}} \big|_{x_0 = z_1 - z_2, x_2 = z_2} \\
&\quad = \langle e^{x_0 L(1)} w'_{(a_3)}, \mathcal{Y}(w_{(a_1)}, e^{-\pi i} x_0) \Omega(Y_{a_2})(w_{(a_2)}, x_2) \mathbf{1} \rangle_{W^{a_3}} \big|_{x_0 = z_1 - z_2, x_2 = z_2},
\end{aligned} \tag{3.7}$$

where Y_{a_2} is the vertex operator for the W^e-module W^{a_2}. Since $\Omega(Y_{a_2}) \in \mathcal{V}_{a_2 e}^{a_2}$, by commutativity, there exist $\mathcal{Y}_1 \in \mathcal{V}_{a_2 a_1}^{a_3}$ and $\mathcal{Y}_2 \in \mathcal{V}_{a_1 e}^{a_1}$, such that the (multivalued) analytic function in the last line of (3.7) defined on the region $|z_1 - z_2| > |z_2| > 0$ and the (multivalued) analytic function

$$\langle e^{x_0 L(1)} w'_{(a_3)}, \mathcal{Y}_1(w_{(a_2)}, x_2) \mathcal{Y}_2(w_{(a_1)}, e^{-\pi i} x_0) \mathbf{1} \rangle_{W^{a_3}} \big|_{x_0 = z_1 - z_2, x_2 = z_2} \tag{3.8}$$

defined on the region $|z_2| > |z_1 - z_2| > 0$ are analytic extensions of each other. Moreover, by associativity, there exist $\mathcal{Y}_3 \in \mathcal{V}_{a_2 a_1}^{a_3}$ and $\mathcal{Y}_4 \in \mathcal{V}_{a_3 e}^{a_3}$, such that the (multivalued) analytic function (3.8) defined on the region $|z_2| > |z_1 - z_2| > 0$ and the (multivalued) analytic function

$$\langle e^{x_0 L(1)} w'_{(a_3)}, \mathcal{Y}_4(\mathcal{Y}_3(w_{(a_2)}, x_1) w_{(a_1)}, e^{-\pi i} x_0) \mathbf{1} \rangle_{W^{a_3}} \big|_{x_0 = z_1 - z_2, x_1 = z_1} \tag{3.9}$$

defined on the region $|z_1 - z_2| > |z_1| > 0$ are equal on the intersection $|z_2| > |z_1 - z_2| > |z_1| > 0$. Since $\Omega^{-1}(\mathcal{Y}_4) \in \mathcal{V}_{e a_3}^{a_3}$, we have $\Omega^{-1}(\mathcal{Y}_4) = s Y_{a_3}$ for some scalar $s \in \mathbb{C}$, where Y_{a_3} is the vertex operator for the W^e-module W^{a_3}. So we see that when $|z_1 - z_2| > |z_1| > 0$,

$$\begin{aligned}
&\langle e^{x_0 L(1)} w'_{(a_3)}, \mathcal{Y}_4(\mathcal{Y}_3(w_{(a_2)}, x_1) w_{(a_1)}, e^{-\pi i} x_0) \mathbf{1} \rangle_{W^{a_3}} \big|_{x_0 = z_1 - z_2, x_1 = z_1} \\
&\quad = \langle w'_{(a_3)}, e^{x_0 L(-1)} \mathcal{Y}_4(\mathcal{Y}_3(w_{(a_2)}, x_1) w_{(a_1)}, e^{-\pi i} x_0) \mathbf{1} \rangle_{W^{a_3}} \big|_{x_0 = z_1 - z_2, x_1 = z_1} \\
&\quad = \langle w'_{(a_3)}, \Omega^{-1}(\mathcal{Y}_4)(\mathbf{1}, x_0) \mathcal{Y}_3(w_{(a_2)}, x_1) w_{(a_1)} \rangle_{W^{a_3}} \big|_{x_0 = z_1 - z_2, x_1 = z_1} \\
&\quad = \langle w'_{(a_3)}, s \mathcal{Y}_3(w_{(a_2)}, x_1) w_{(a_1)} \rangle_{W^{a_3}} \big|_{x_1 = z_1}.
\end{aligned} \tag{3.10}$$

So

$$\langle w'_{(a_3)}, \Omega(\mathcal{Y})(w_{(a_2)}, x_1) w_{(a_1)} \rangle_{W^{a_3}} \big|_{x_1 = z_1} = \langle w'_{(a_3)}, s \mathcal{Y}_3(w_{(a_2)}, x_1) w_{(a_1)} \rangle_{W^{a_3}} \big|_{x_1 = z_1} \tag{3.11}$$

as multivalued analytic functions in z_1 for $w_{(a_1)} \in W^{a_1}$, $w_{(a_2)} \in W^{a_2}$ and $w'_{(a_3)} \in (W^{a_3})'$. Thus there exists $p \in \mathbb{Z}$ such that

$$\begin{aligned}
&\langle w'_{(a_3)}, \Omega(\mathcal{Y})(w_{(a_2)}, x_1)w_{(a_1)}\rangle_{W^{a_3}}\big|_{x_1^n=e^{n\log z_1}}\\
&\quad= \langle w'_{(a_3)}, s\mathcal{Y}_3(w_{(a_2)}, x_1)w_{(a_1)}\rangle_{W^{a_3}}\big|_{x_1^n=e^{n(\log z_1+2\pi p i)}}.
\end{aligned} \tag{3.12}$$

By the weight condition (see Definition 2.3), the second line of (3.12) is proportional to

$$\langle w'_{(a_3)}, \mathcal{Y}_3(w_{(a_2)}, x_1)w_{(a_1)}\rangle_{W^{a_3}}\big|_{x_1^n=e^{n\log z_1}}.$$

Thus

$$\Omega(\mathcal{Y}) = s'\mathcal{Y}_3 \in \mathcal{V}_{a_2a_1}^{a_3} \tag{3.13}$$

for some scalar $s' \in \mathbb{C}$. Since $\mathcal{Y} \in \mathcal{V}_{a_1a_2}^{a_3}$ is arbitrary, we have

$$\Omega(\mathcal{V}_{a_1a_2}^{a_3}) \subset \mathcal{V}_{a_2a_1}^{a_3}. \tag{3.14}$$

Moreover, since $a_1, a_2, a_3 \in \mathcal{A}$ are arbitrary, we also have

$$\Omega(\mathcal{V}_{a_2a_1}^{a_3}) \subset \mathcal{V}_{a_1a_2}^{a_3}. \tag{3.15}$$

From the defintion of Ω and the weight condition, for any $\mathcal{Y}' \in \mathcal{V}_{a_2a_1}^{a_3}$, $\Omega^2(\mathcal{Y}') = a\mathcal{Y}' \in \mathcal{V}_{a_2a_1}^{a_3}$ with some nonzero scalar $a \in \mathbb{C}$. So the restriction of Ω to $\mathcal{V}_{a_1a_2}^{a_3}$ is surjective. Since Ω itself is an isomorphism, we see that the restriction of Ω to $\mathcal{V}_{a_1a_2}^{a_3}$ is injective. Therefore the restriction of Ω to $\mathcal{V}_{a_1a_2}^{a_3}$ is an isomorphism from $\mathcal{V}_{a_1a_2}^{a_3}$ to $\mathcal{V}_{a_2a_1}^{a_3}$. ∎

Now we derive the relations between the duality properties and the Jacobi identity. It was proved in [H9] that the Jacobi identity follows from the generalized rationality, commutativity and associativity properties of intertwining operator algebras. Conversely, we have

Theorem 3.3. *In the presence of the axioms for intertwining operator algebras except for associativity and skew-symmetry, the generalized rationality, commutativity and associativity follow from the Jacobi identity.*

Proof. Fix any $a_1, a_2, a_3, a_4 \in \mathcal{A}$, $w_{(a_1)} \in W^{a_1}$, $w_{(a_2)} \in W^{a_2}$, $w_{(a_3)} \in W^{a_3}$, $w'_{(a_4)} \in (W^{a_4})'$, $\mathcal{Z} \in \coprod_{a_5\in\mathcal{A}} \mathcal{V}_{a_1a_5}^{a_4} \otimes \mathcal{V}_{a_2a_3}^{a_5}$ and $\alpha \in \mathbb{A}(a_1, a_2, a_3, a_4)$. Then from (2.127), (2.128), (2.129) and Res_{x_0} of both sides of (2.137), we have

$$\begin{aligned}
&\langle w'_{(a_4)}, f_\alpha^{a_1,a_2,a_3,a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, [\mathcal{Z}]_P; x_1, x_2)\rangle_{W^{a_4}}\\
&\qquad-\langle w'_{(a_4)}, g_\alpha^{a_1,a_2,a_3,a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, \mathcal{B}([\mathcal{Z}]_P); x_1, x_2)\rangle_{W^{a_4}}\\
&\quad= \mathrm{Res}_{x_0}x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\langle w'_{(a_4)}, h_\alpha^{a_1,a_2,a_3,a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, \mathcal{F}([\mathcal{Z}]_P); x_0, x_2)\rangle_{W^{a_4}}.
\end{aligned} \tag{3.16}$$

Since

$$x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)$$

contains only terms of positive powers of x_0, the right-hand side of (3.16) involves only the part

$$\left(\langle w'_{(a_4)}, h_\alpha^{a_1,a_2,a_3,a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, \mathcal{F}([\mathcal{Z}]_P); x_0, x_2)\rangle_{W^{a_4}}\right)^-$$

of

$$\langle w'_{(a_4)}, h_\alpha^{a_1,a_2,a_3,a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, \mathcal{F}([\mathcal{Z}]_P); x_0, x_2)\rangle_{W^{a_4}}$$

containing only negative powers of x_0. We know that

$$\langle w'_{(a_4)}, h_\alpha^{a_1,a_2,a_3,a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, \mathcal{F}([\mathcal{Z}]_P); x_0, x_2)\rangle_{W^{a_4}}$$

has only finitely many terms in negative powers of x_0. Hence only finitely many powers of x_0 appears in

$$\left(\langle w'_{(a_4)}, h_\alpha^{a_1,a_2,a_3,a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, \mathcal{F}([\mathcal{Z}]_P); x_0, x_2)\rangle_{W^{a_4}}\right)^- .$$

In particular, both

$$\begin{aligned}
&\left(\langle w'_{(a_4)}, h_\alpha^{a_1,a_2,a_3,a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, \mathcal{F}([\mathcal{Z}]_P); x_1-x_2, x_2)\rangle_{W^{a_4}}\right)^- \\
&\quad = \left(\langle w'_{(a_4)}, h_\alpha^{a_1,a_2,a_3,a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, \mathcal{F}([\mathcal{Z}]_P); x_0, x_2)\rangle_{W^{a_4}}\right)^- \Big|_{x_0=x_1-x_2}
\end{aligned}$$

and

$$\begin{aligned}
&\left(\langle w'_{(a_4)}, h_\alpha^{a_1,a_2,a_3,a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, \mathcal{F}([\mathcal{Z}]_P); -x_2+x_1, x_2)\rangle_{W^{a_4}}\right)^- \\
&\quad = \left(\langle w'_{(a_4)}, h_\alpha^{a_1,a_2,a_3,a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, \mathcal{F}([\mathcal{Z}]_P); x_0, x_2)\rangle_{W^{a_4}}\right)^- \Big|_{x_0=-x_2+x_1}
\end{aligned}$$

are well defined. Thus the right-hand side of (3.16) is equal to

$$\begin{aligned}
&\mathrm{Res}_{x_0} x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\left(\langle w'_{(a_4)}, h_\alpha^{a_1,a_2,a_3,a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, \mathcal{F}([\mathcal{Z}]_P); x_0, x_2)\rangle_{W^{a_4}}\right)^- \\
&\quad = \left(\langle w'_{(a_4)}, h_\alpha^{a_1,a_2,a_3,a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, \mathcal{F}([\mathcal{Z}]_P); x_1-x_2, x_2)\rangle_{W^{a_4}}\right)^- \\
&\qquad - \left(\langle w'_{(a_4)}, h_\alpha^{a_1,a_2,a_3,a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, \mathcal{F}([\mathcal{Z}]_P); -x_2+x_1, x_2)\rangle_{W^{a_4}}\right)^- \\
&\quad = (\iota_{12}-\iota_{21})\left(\frac{\varphi_\alpha(x_1,x_2)}{x_2^r(x_1-x_2)^s}\right) \qquad (3.17)
\end{aligned}$$

for some $\varphi_\alpha(x_1,x_2) \in \mathbb{C}[x_1,x_2]$ and $r, s \in \mathbb{N}$. So we have

$$\begin{aligned}
&\langle w'_{(a_4)}, f_\alpha^{a_1,a_2,a_3,a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, [\mathcal{Z}]_P; x_1, x_2)\rangle_{W^{a_4}} - \iota_{12}\left(\frac{\varphi_\alpha(x_1,x_2)}{x_2^r(x_1-x_2)^s}\right) \\
&\quad = \langle w'_{(a_4)}, g_\alpha^{a_1,a_2,a_3,a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, \mathcal{B}([\mathcal{Z}]_P); x_1, x_2)\rangle_{W^{a_4}} - \iota_{21}\left(\frac{\varphi_\alpha(x_1,x_2)}{x_2^r(x_1-x_2)^s}\right).
\end{aligned} \qquad (3.18)$$

The left hand side of (3.18) involves only finitely many negative powers of x_2, and the right hand side of (3.18) involves only finitely many positive powers of x_2. Thus each side of the above equation involves only finitely many powers of x_2. Moreover, the coefficient of each power of x_2 on either side of (3.18) is a Laurent polynomial in x_1. So both sides of (3.18) are equal to a Laurent polynomial $\psi(x_1x_2) \in \mathbb{C}[x_1, x_1^{-1}, x_2, x_2^{-1}]$. Define

$$\begin{aligned} F_\alpha : \quad & (W^{a_4})' \otimes W^{a_1} \otimes W^{a_2} \otimes W^{a_3} \otimes \pi_P(\coprod_{a_5 \in \mathcal{A}} \mathcal{V}_{a_1 a_5}^{a_4} \otimes \mathcal{V}_{a_2 a_3}^{a_5}) \\ & \qquad \longrightarrow \mathbb{C}[x_1, x_1^{-1}, x_2, x_2^{-1}, (x_1 - x_2)^{-1}] \end{aligned} \tag{3.19}$$

by

$$F_\alpha(w'_{(a_4)}, w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, [\mathcal{Z}]_P; x_1, x_2) = \frac{\varphi_\alpha(x_1, x_2)}{x_2^r (x_1 - x_2)^s} + \psi(x_1, x_2), \tag{3.20}$$

where $F_\alpha(w'_{(a_4)}, w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, [\mathcal{Z}]_P; x_1, x_2)$ is the image of

$$w'_{(a_4)} \otimes w_{(a_1)} \otimes w_{(a_2)} \otimes w_{(a_3)} \otimes [\mathcal{Z}]_P$$

under F_α. Then we have

$$\begin{aligned} & \langle w'_{(a_4)}, f_\alpha^{a_1, a_2, a_3, a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, [\mathcal{Z}]_P; x_1, x_2) \rangle_{W^{a_4}} \\ & \qquad = \iota_{12} F_\alpha(w'_{(a_4)}, w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, [\mathcal{Z}]_P; x_1, x_2) \end{aligned} \tag{3.21}$$

and

$$\begin{aligned} & \langle w'_{(a_4)}, g_\alpha^{a_1, a_2, a_3, a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, \mathcal{B}([\mathcal{Z}]_P); x_1, x_2) \rangle_{W^{a_4}} \\ & \qquad = \iota_{21} F_\alpha(w'_{(a_4)}, w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, [\mathcal{Z}]_P; x_1, x_2) \end{aligned} \tag{3.22}$$

Thus the generalized rationality of products and commutativity hold.

On the other hand, from (2.127), (2.128), (2.129) and Res_{x_1} of both sides of (2.137), using the same argument as in the proof of (3.18) above, we obtain

$$\begin{aligned} & \langle w'_{(a_4)}, f_\alpha^{a_1, a_2, a_3, a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, [\mathcal{Z}]_P; x_0 + x_2, x_2) \rangle_{W^{a_4}} \\ & \quad - \langle w'_{(a_4)}, h_\alpha^{a_1, a_2, a_3, a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, \mathcal{F}([\mathcal{Z}]_P); x_0, x_2) \rangle_{W^{a_4}} \\ & = \big(\langle w'_{(a_4)}, g_\alpha^{a_1, a_2, a_3, a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, \mathcal{B}([\mathcal{Z}]_P); x_0 + x_2, x_2) \rangle_{W^{a_4}} \big)^- \\ & \quad - \big(\langle w'_{(a_4)}, g_\alpha^{a_1, a_2, a_3, a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, \mathcal{B}([\mathcal{Z}]_P); x_2 + x_0, x_2) \rangle_{W^{a_4}} \big)^- \\ & = (\iota_{02} - \iota_{20}) \left(\frac{\phi_\alpha(x_0, x_2)}{x_2^{r'} (x_0 + x_2)^{s'}} \right) \end{aligned} \tag{3.23}$$

for some $\phi_\alpha(x_0, x_2) \in \mathbb{C}[x_0, x_2]$ and $r', s' \in \mathbb{N}$, where

$$\big(\langle w'_{(a_4)}, g_\alpha^{a_1, a_2, a_3, a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, \mathcal{B}([\mathcal{Z}]_P); x_1, x_2) \rangle_{W^{a_4}} \big)^-$$

is the part of

$$\langle w'_{(a_4)}, g_\alpha^{a_1,a_2,a_3,a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, \mathcal{B}([\mathcal{Z}]_P); x_1, x_2)\rangle_{W^{a_4}}$$

containing only the terms in negative powers of x_1,

$$\begin{aligned}&\big(\langle w'_{(a_4)}, g_\alpha^{a_1,a_2,a_3,a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, \mathcal{B}([\mathcal{Z}]_P); x_0+x_2, x_2)\rangle_{W^{a_4}}\big)^-\\&\quad= \big(\langle w'_{(a_4)}, g_\alpha^{a_1,a_2,a_3,a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, \mathcal{B}([\mathcal{Z}]_P); x_1, x_2)\rangle_{W^{a_4}}\big)^-\Big|_{x_1=x_0+x_2}\end{aligned}$$

and

$$\begin{aligned}&\big(\langle w'_{(a_4)}, g_\alpha^{a_1,a_2,a_3,a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, \mathcal{B}([\mathcal{Z}]_P); x_2+x_0, x_2)\rangle_{W^{a_4}}\big)^-\\&\quad= \big(\langle w'_{(a_4)}, g_\alpha^{a_1,a_2,a_3,a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, \mathcal{B}([\mathcal{Z}]_P); x_1, x_2)\rangle_{W^{a_4}}\big)^-\Big|_{x_1=x_2+x_0}.\end{aligned}$$

So we have

$$\begin{aligned}&\langle w'_{(a_4)}, f_\alpha^{a_1,a_2,a_3,a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, [\mathcal{Z}]_P; x_0+x_2, x_2)\rangle_{W^{a_4}} - \iota_{02}\left(\frac{\phi_\alpha(x_0,x_2)}{x_2^{r'}(x_0+x_2)^{s'}}\right)\\&\quad= \langle w'_{(a_4)}, h_\alpha^{a_1,a_2,a_3,a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, \mathcal{F}([\mathcal{Z}]_P); x_0, x_2)\rangle_{W^{a_4}} - \iota_{20}\left(\frac{\phi_\alpha(x_0,x_2)}{x_2^{r'}(x_0+x_2)^{s'}}\right).\end{aligned} \tag{3.24}$$

Moreover, replacing x_1 by x_0+x_2 and then expanding powers of x_0+x_2 in nonnegative powers of x_2 on both sides of (3.21), we obtain

$$\begin{aligned}&\langle w'_{(a_4)}, f_\alpha^{a_1,a_2,a_3,a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, [\mathcal{Z}]_P; x_0+x_2, x_2)\rangle_{W^{a_4}}\\&\quad= \iota_{02}((\iota_{12}F_\alpha(w'_{(a_4)}, w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, [\mathcal{Z}]_P; x_1, x_2))|_{x_1=x_0+x_2}).\end{aligned} \tag{3.25}$$

But for any element $\phi(x_1,x_2) \in \mathbb{C}[x_1, x^{-1}, x_2, x_2^{-1}, (x_1-x_2)^{-1}]$, we have

$$\iota_{02}((\iota_{12}\phi(x_1,x_2))|_{x_1=x_0+x_2}) = \iota_{02}\phi(x_0+x_2, x_2).$$

Hence from (3.25), we obtain

$$\begin{aligned}&\langle w'_{(a_4)}, f_\alpha^{a_1,a_2,a_3,a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, [\mathcal{Z}]_P; x_0+x_2, x_2)\rangle_{W^{a_4}}\\&\quad= \iota_{02}F_\alpha(w'_{(a_4)}, w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, [\mathcal{Z}]_P; x_0+x_2, x_2)).\end{aligned} \tag{3.26}$$

So the left hand side of (3.24) involves only finitely many negative powers of x_2, and the right hand side of (3.24) involves only finitely many positive powers of x_2. Thus each side of (3.24) involves only finitely many powers of x_2. Moreover, the coefficient of each power of

x_2 on either side of (3.24) is a Laurent polynomial in x_0. So both hand sides of (3.24) are equal to a Laurent polynomial $\tau(x_0, x_2) \in \mathbb{C}[x_0, x_0^{-1}, x_2, x_2^{-1}]$. Thus

$$\begin{aligned}
&\langle w'_{(a_4)}, f_\alpha^{a_1,a_2,a_3,a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, [\mathcal{Z}]_P; x_0 + x_2, x_2)\rangle_{W^{a_4}} \\
&\quad = \iota_{02}\Big(\frac{\phi_\alpha(x_0, x_2)}{x_2^{r'}(x_0 + x_2)^{s'}} + \tau(x_0, x_2)\Big). \qquad (3.27)
\end{aligned}$$

Comparing (3.27) with (3.21), we see that

$$\frac{\phi_\alpha(x_0, x_2)}{x_2^{r'}(x_0 + x_2)^{s'}} + \tau(x_0, x_2) = F_\alpha(w'_{(a_4)}, w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, [\mathcal{Z}]_P; x_0 + x_2, x_2). \qquad (3.28)$$

So

$$\begin{aligned}
&\langle w'_{(a_4)}, h_\alpha^{a_1,a_2,a_3,a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, \mathcal{F}([\mathcal{Z}]_P); x_0, x_2)\rangle_{W^{a_4}} \\
&\quad = \iota_{20}\Big(\frac{\phi_\alpha(x_0, x_2)}{x_2^{r'}(x_0 + x_2)^{s'}} + \tau(x_0, x_2)\Big) \\
&\quad = \iota_{20} F_\alpha(w'_{(a_4)}, w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, [\mathcal{Z}]_P; x_2 + x_0, x_2). \qquad (3.29)
\end{aligned}$$

Therefore the generalized rationality of iterates and associativity hold. ∎

Moreover, we shall derive the locality.

Theorem 3.4 (Locality). *In the presence of the axioms for intertwining operator algebras except for associativity and skew-symmetry, we assume that the Jacobi identity holds. Then for* $a_1, a_2, a_3, a_4 \in \mathcal{A}$, $w_{(a_1)} \in W^{a_1}$, $w_{(a_2)} \in W^{a_2}$, $w_{(a_3)} \in W^{a_3}$, $w'_{(a_4)} \in (W^{a_4})'$, *and*

$$\mathcal{Z} \in \coprod_{a_5 \in \mathcal{A}} \mathcal{V}_{a_1 a_5}^{a_4} \otimes \mathcal{V}_{a_2 a_3}^{a_5} \subset \coprod_{a_1,a_2,a_3,a_4,a_5 \in \mathcal{A}} \mathcal{V}_{a_1 a_5}^{a_4} \otimes \mathcal{V}_{a_2 a_3}^{a_5}, \qquad (3.30)$$

$\alpha \in \mathbb{A}(a_1, a_2, a_3, a_4)$, *there exist* $n_1, n_2 \in \mathbb{N}$ *such that the following equations (locality) hold :*

$$\begin{aligned}
&(x_1 - x_2)^{n_1}\langle w'_{(a_4)}, f_\alpha^{a_1,a_2,a_3,a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, [\mathcal{Z}]_P; x_1, x_2)\rangle_{W^{a_4}} \\
&\quad = (x_1 - x_2)^{n_1}\langle w'_{(a_4)}, g_\alpha^{a_1,a_2,a_3,a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, \mathcal{B}([\mathcal{Z}]_P); x_1, x_2)\rangle_{W^{a_4}}, \qquad (3.31)
\end{aligned}$$

$$\begin{aligned}
&(x_0 + x_2)^{n_2}\langle w'_{(a_4)}, f_\alpha^{a_1,a_2,a_3,a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, [\mathcal{Z}]_P; x_0 + x_2, x_2)\rangle_{W^{a_4}} \\
&\quad = (x_0 + x_2)^{n_2}\langle w'_{(a_4)}, h_\alpha^{a_1,a_2,a_3,a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, \mathcal{F}([\mathcal{Z}]_P); x_0, x_2)\rangle_{W^{a_4}}, \qquad (3.32)
\end{aligned}$$

where $f_\alpha^{a_1,a_2,a_3,a_4}$, $g_\alpha^{a_1,a_2,a_3,a_4}$, $h_\alpha^{a_1,a_2,a_3,a_4}$ *are the linear maps given in (2.127), (2.128) and (2.129) satisfying the relations from (2.131) to (2.136), respectively.*

Proof. By (3.18) and (3.24), we see that the equations (locality) (3.31) and (3.32) hold. ∎

Theorem 3.5. *In the presence of the axioms for intertwining operator algebras except for associativity and skew-symmetry, the Jacobi identity and the locality property are equivalent.*

Proof. By Theorem 3.4, we only need to derive Jacobi identity from the locality property. Fix any $a_1, a_2, a_3, a_4 \in \mathcal{A}$, $w_{(a_1)} \in W^{a_1}$, $w_{(a_2)} \in W^{a_2}$, $w_{(a_3)} \in W^{a_3}$, $w'_{(a_4)} \in (W^{a_4})'$, and any

$$\mathcal{Z} \in \coprod_{a_5 \in \mathcal{A}} \mathcal{V}_{a_1 a_5}^{a_4} \otimes \mathcal{V}_{a_2 a_3}^{a_5} \subset \coprod_{a_1,a_2,a_3,a_4,a_5 \in \mathcal{A}} \mathcal{V}_{a_1 a_5}^{a_4} \otimes \mathcal{V}_{a_2 a_3}^{a_5}. \tag{3.33}$$

By (3.31) and (3.32) we get

$$\begin{aligned}
&\langle w'_{(a_4)}, x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) f_\alpha^{a_1,a_2,a_3,a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, [\mathcal{Z}]_P; x_1, x_2)\\
&\qquad - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) g_\alpha^{a_1,a_2,a_3,a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, \mathcal{B}([\mathcal{Z}]_P); x_1, x_2)\rangle_{W^{a_4}}\\
&= \langle w'_{(a_4)}, x_0^{-n_1-1}\delta\left(\frac{x_1 - x_2}{x_0}\right)(x_1 - x_2)^{n_1} f_\alpha^{a_1,a_2,a_3,a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, [\mathcal{Z}]_P; x_1, x_2)\\
&\qquad - x_0^{-n_1-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right)(x_1 - x_2)^{n_1} g_\alpha^{a_1,a_2,a_3,a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, \mathcal{B}([\mathcal{Z}]_P); x_1, x_2)\rangle_{W^{a_4}}\\
&= \langle w'_{(a_4)}, x_0^{-n_1}(x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right))\\
&\qquad \cdot (x_1 - x_2)^{n_1} f_\alpha^{a_1,a_2,a_3,a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, [\mathcal{Z}]_P; x_1, x_2)\rangle_{W^{a_4}}\\
&= \langle w'_{(a_4)}, x_0^{-n_1} x_1^{-1}\delta\left(\frac{x_2 + x_0}{x_1}\right)(x_1 - x_2)^{n_1} f_\alpha^{a_1,a_2,a_3,a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, [\mathcal{Z}]_P; x_1, x_2)\rangle_{W^{a_4}}\\
&= \langle w'_{(a_4)}, x_1^{-1}\delta\left(\frac{x_2 + x_0}{x_1}\right) f_\alpha^{a_1,a_2,a_3,a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, [\mathcal{Z}]_P; x_2 + x_0, x_2)\rangle_{W^{a_4}}\\
&= \langle w'_{(a_4)}, x_1^{-n_2-1}\delta\left(\frac{x_2 + x_0}{x_1}\right)(x_0 + x_2)^{n_2}\\
&\qquad \cdot f_\alpha^{a_1,a_2,a_3,a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, [\mathcal{Z}]_P; x_2 + x_0, x_2)\rangle_{W^{a_4}}\\
&= \langle w'_{(a_4)}, x_1^{-n_2-1}\delta\left(\frac{x_2 + x_0}{x_1}\right)(x_0 + x_2)^{n_2}\\
&\qquad \cdot h_\alpha^{a_1,a_2,a_3,a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, \mathcal{F}([\mathcal{Z}]_P); x_0, x_2)\rangle_{W^{a_4}}\\
&= \langle w'_{(a_4)}, x_1^{-1}\delta\left(\frac{x_2 + x_0}{x_1}\right) h_\alpha^{a_1,a_2,a_3,a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, \mathcal{F}([\mathcal{Z}]_P); x_0, x_2)\rangle_{W^{a_4}}\\
&= \langle w'_{(a_4)}, x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) h_\alpha^{a_1,a_2,a_3,a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, \mathcal{F}([\mathcal{Z}]_P); x_0, x_2)\rangle_{W^{a_4}}
\end{aligned} \tag{3.34}$$

for $\alpha \in \mathbb{A}(a_1, a_2, a_3, a_4)$. Since $w'_{(a_4)} \in (W^{a_4})'$ is arbitrary, we obtain the Jacobi identity

$$
\begin{aligned}
x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right) & f_\alpha^{a_1,a_2,a_3,a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, [\mathcal{Z}]_P; x_1, x_2) \\
& -x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right) g_\alpha^{a_1,a_2,a_3,a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, \mathcal{B}([\mathcal{Z}]_P); x_1, x_2) \\
& = x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right) h_\alpha^{a_1,a_2,a_3,a_4}(w_{(a_1)}, w_{(a_2)}, w_{(a_3)}, \mathcal{F}([\mathcal{Z}]_P); x_0, x_2)
\end{aligned}
\tag{3.35}
$$

for $\alpha \in \mathbb{A}(a_1, a_2, a_3, a_4)$. ∎

4 Genus-zero Moore-Seiberg equation

In this section, we derive the genus-zero Moore-Seiberg equations from the convergence property, associativity, commutativity and skew-symmetry.

The skew-symmetry, fusing and braiding isomorphisms induce isomorphisms between vector spaces containing the domains and images of these isomorphisms. These induced isomorphisms are not independent. To describe the relations satisfied by these induced isomorphisms, we need to introduce notations for certain particular induced isomorphisms.

Firstly, we shall define five linear maps, which respectively correspond to the multiplications, iterates and the mixtures of the two operations of three intertwining operators. The first one is

$$
\mathbf{PP} : \coprod_{a_1,\cdots,a_7\in\mathcal{A}} \mathcal{V}_{a_1a_6}^{a_5} \otimes \mathcal{V}_{a_2a_7}^{a_6} \otimes \mathcal{V}_{a_3a_4}^{a_7} \longrightarrow (\mathrm{Hom}(W\otimes W\otimes W\otimes W, W))\{x_1, x_2, x_3\} \tag{4.1}
$$

defined using products of intertwining operators as follows: For

$$
\mathcal{Z} \in \coprod_{a_1,\cdots,a_7\in\mathcal{A}} \mathcal{V}_{a_1a_6}^{a_5} \otimes \mathcal{V}_{a_2a_7}^{a_6} \otimes \mathcal{V}_{a_3a_4}^{a_7}, \tag{4.2}
$$

the element $\mathbf{PP}(\mathcal{Z})$ to be defined can also be viewed as a map from $W\otimes W\otimes W\otimes W$ to $W\{x_1, x_2, x_3\}$. We define $\mathbf{PP}$ by linearity and by

$$
\begin{aligned}
&(\mathbf{PP}(\mathcal{Y}_1\otimes\mathcal{Y}_2\otimes\mathcal{Y}_3))(w_{(a_{i_1})}, w_{(a_{i_2})}, w_{(a_{i_3})}, w_{(a_{i_4})}; x_1, x_2, x_3) \\
&\quad = \begin{cases} \mathcal{Y}_1(w_{(a_{i_1})}, x_1)\mathcal{Y}_2(w_{(a_{i_2})}, x_2)\mathcal{Y}_3(w_{(a_{i_3})}, x_3)w_{(a_{i_4})}, & i_1=1, i_2=2, i_3=3, i_4=4, \\ 0, & \text{otherwise} \end{cases}
\end{aligned}
\tag{4.3}
$$

for $a_1, \ldots, a_7, a_{i_1}, \cdots, a_{i_4} \in \mathcal{A}$, $\mathcal{Y}_1 \in \mathcal{V}_{a_1a_6}^{a_5}$, $\mathcal{Y}_2 \in \mathcal{V}_{a_2a_7}^{a_6}$, $\mathcal{Y}_3 \in \mathcal{V}_{a_3a_4}^{a_7}$, and $w_{(a_{i_k})} \in W^{a_{i_k}}$ with $k = 1, \cdots, 4$. Then we have an isomorphism

$$
\widetilde{\mathbf{PP}} : \frac{\coprod_{a_1,\cdots,a_7\in\mathcal{A}} \mathcal{V}_{a_1a_6}^{a_5} \otimes \mathcal{V}_{a_2a_7}^{a_6} \otimes \mathcal{V}_{a_3a_4}^{a_7}}{\mathrm{Ker}\ \mathbf{PP}} \longrightarrow \mathbf{PP}\left(\coprod_{a_1,\cdots,a_7\in\mathcal{A}} \mathcal{V}_{a_1a_6}^{a_5} \otimes \mathcal{V}_{a_2a_7}^{a_6} \otimes \mathcal{V}_{a_3a_4}^{a_7}\right) \tag{4.4}
$$

such that the following diagram commute:

$$\begin{tikzcd}
\coprod_{a_1,\cdots,a_7\in\mathcal{A}} \mathcal{V}_{a_1a_6}^{a_5}\otimes\mathcal{V}_{a_2a_7}^{a_6}\otimes\mathcal{V}_{a_3a_4}^{a_7} \arrow[r, "\mathbf{PP}"] \arrow[d, "\pi"'] & \mathbf{PP}\left(\coprod_{a_1,\cdots,a_7\in\mathcal{A}} \mathcal{V}_{a_1a_6}^{a_5}\otimes\mathcal{V}_{a_2a_7}^{a_6}\otimes\mathcal{V}_{a_3a_4}^{a_7}\right) \\
\dfrac{\coprod_{a_1,\cdots,a_7\in\mathcal{A}} \mathcal{V}_{a_1a_6}^{a_5}\otimes\mathcal{V}_{a_2a_7}^{a_6}\otimes\mathcal{V}_{a_3a_4}^{a_7}}{\mathrm{Ker}\,\mathbf{PP}} \arrow[ur, "\widetilde{\mathbf{PP}}"] &
\end{tikzcd} \tag{4.5}$$

where π is the corresponding canonical projection map.

The second one is

$$\mathbf{IP}: \coprod_{a_1,\cdots,a_7\in\mathcal{A}} \mathcal{V}_{a_1a_2}^{a_6}\otimes\mathcal{V}_{a_6a_7}^{a_5}\otimes\mathcal{V}_{a_3a_4}^{a_7} \longrightarrow (\mathrm{Hom}(W\otimes W\otimes W\otimes W, W))\{x_0,x_2,x_3\} \tag{4.6}$$

defined using products and iterates of intertwining operators as follows: For

$$\mathcal{Z}\in \coprod_{a_1,\cdots,a_7\in\mathcal{A}} \mathcal{V}_{a_1a_2}^{a_6}\otimes\mathcal{V}_{a_6a_7}^{a_5}\otimes\mathcal{V}_{a_3a_4}^{a_7}, \tag{4.7}$$

the element $\mathbf{IP}(\mathcal{Z})$ to be defined can also be viewed as a map from $W\otimes W\otimes W\otimes W$ to $W\{x_0,x_2,x_3\}$. Then we define $\mathbf{IP}$ by linearity and by

$$\begin{aligned}
&(\mathbf{IP}(\mathcal{Y}_1\otimes\mathcal{Y}_2\otimes\mathcal{Y}_3))(w_{(a_{i_1})},w_{(a_{i_2})},w_{(a_{i_3})},w_{(a_{i_4})};x_0,x_2,x_3)\\
&\quad=\begin{cases}\mathcal{Y}_2(\mathcal{Y}_1(w_{(a_{i_1})},x_0)w_{(a_{i_2})},x_2)\mathcal{Y}_3(w_{(a_{i_3})},x_3)w_{(a_{i_4})}, & i_1=1,i_2=2,i_3=3,i_4=4,\\ 0, & \text{otherwise}\end{cases}
\end{aligned} \tag{4.8}$$

for $a_1,\ldots,a_7,a_{i_1},\cdots,a_{i_4}\in\mathcal{A}$, $\mathcal{Y}_1\in\mathcal{V}_{a_1a_2}^{a_6}$, $\mathcal{Y}_2\in\mathcal{V}_{a_6a_7}^{a_5}$, $\mathcal{Y}_3\in\mathcal{V}_{a_3a_4}^{a_7}$, and $w_{(a_{i_k})}\in W^{a_{i_k}}$ with $k=1,\cdots,4$. Then we have an isomorphism

$$\widetilde{\mathbf{IP}}: \frac{\coprod_{a_1,\cdots,a_7\in\mathcal{A}} \mathcal{V}_{a_1a_2}^{a_6}\otimes\mathcal{V}_{a_6a_7}^{a_5}\otimes\mathcal{V}_{a_3a_4}^{a_7}}{\mathrm{Ker}\,\mathbf{IP}} \longrightarrow \mathbf{IP}\left(\coprod_{a_1,\cdots,a_7\in\mathcal{A}} \mathcal{V}_{a_1a_2}^{a_6}\otimes\mathcal{V}_{a_6a_7}^{a_5}\otimes\mathcal{V}_{a_3a_4}^{a_7}\right). \tag{4.9}$$

The third one is

$$\mathbf{I^P}: \coprod_{a_1,\cdots,a_7\in\mathcal{A}} \mathcal{V}_{a_1a_7}^{a_6}\otimes\mathcal{V}_{a_2a_3}^{a_7}\otimes\mathcal{V}_{a_6a_4}^{a_5} \longrightarrow (\mathrm{Hom}(W\otimes W\otimes W\otimes W, W))\{y_1,y_2,x_3\} \tag{4.10}$$

defined using products and iterates of intertwining operators as follows: For

$$\mathcal{Z}\in \coprod_{a_1,\cdots,a_7\in\mathcal{A}} \mathcal{V}_{a_1a_7}^{a_6}\otimes\mathcal{V}_{a_2a_3}^{a_7}\otimes\mathcal{V}_{a_6a_4}^{a_5}, \tag{4.11}$$

the element $\mathbf{I^P}(\mathcal{Z})$ to be defined can also be viewed as a map from $W \otimes W \otimes W \otimes W$ to $W\{y_1, y_2, x_3\}$. Then we define $\mathbf{I^P}$ by linearity and by

$$
\begin{aligned}
&(\mathbf{I^P}(\mathcal{Y}_1 \otimes \mathcal{Y}_2 \otimes \mathcal{Y}_3))(w_{(a_{i_1})}, w_{(a_{i_2})}, w_{(a_{i_3})}, w_{(a_{i_4})}; y_1, y_2, x_3)\\
&\quad = \begin{cases} \mathcal{Y}_3(\mathcal{Y}_1(w_{(a_{i_1})}, y_1)\mathcal{Y}_2(w_{(a_{i_2})}, y_2)w_{(a_{i_3})}, x_3)w_{(a_{i_4})}, & i_1 = 1, i_2 = 2, i_3 = 3, i_4 = 4, \\ 0, & \text{otherwise} \end{cases}
\end{aligned} \tag{4.12}
$$

for $a_1, \ldots, a_7, a_{i_1}, \cdots, a_{i_4} \in \mathcal{A}$, $\mathcal{Y}_1 \in \mathcal{V}_{a_1 a_7}^{a_6}$, $\mathcal{Y}_2 \in \mathcal{V}_{a_2 a_3}^{a_7}$, $\mathcal{Y}_3 \in \mathcal{V}_{a_6 a_4}^{a_5}$, and $w_{(a_{i_k})} \in W^{a_{i_k}}$ with $k = 1, \cdots, 4$. Then we have an isomorphism

$$
\widetilde{\mathbf{I^P}} : \frac{\coprod_{a_1, \cdots, a_7 \in \mathcal{A}} \mathcal{V}_{a_1 a_7}^{a_6} \otimes \mathcal{V}_{a_2 a_3}^{a_7} \otimes \mathcal{V}_{a_6 a_4}^{a_5}}{\text{Ker } \mathbf{I^P}} \longrightarrow \mathbf{I^P}\left(\coprod_{a_1, \cdots, a_7 \in \mathcal{A}} \mathcal{V}_{a_1 a_7}^{a_6} \otimes \mathcal{V}_{a_2 a_3}^{a_7} \otimes \mathcal{V}_{a_6 a_4}^{a_5}\right). \tag{4.13}
$$

The fourth one is

$$
\mathbf{II} : \coprod_{a_1, \cdots, a_7 \in \mathcal{A}} \mathcal{V}_{a_1 a_2}^{a_7} \otimes \mathcal{V}_{a_7 a_3}^{a_6} \otimes \mathcal{V}_{a_6 a_4}^{a_5} \longrightarrow (\text{Hom}(W \otimes W \otimes W \otimes W, W))\{x_0, y_2, x_3\} \tag{4.14}
$$

defined using iterates of intertwining operators as follows: For

$$
\mathcal{Z} \in \coprod_{a_1, \cdots, a_7 \in \mathcal{A}} \mathcal{V}_{a_1 a_2}^{a_7} \otimes \mathcal{V}_{a_7 a_3}^{a_6} \otimes \mathcal{V}_{a_6 a_4}^{a_5}, \tag{4.15}
$$

the element $\mathbf{II}(\mathcal{Z})$ to be defined can also be viewed as a map from $W \otimes W \otimes W \otimes W$ to $W\{x_0, y_2, x_3\}$. Then we define $\mathbf{II}$ by linearity and by

$$
\begin{aligned}
&(\mathbf{II}(\mathcal{Y}_1 \otimes \mathcal{Y}_2 \otimes \mathcal{Y}_3))(w_{(a_{i_1})}, w_{(a_{i_2})}, w_{(a_{i_3})}, w_{(a_{i_4})}; x_0, y_2, x_3)\\
&\quad = \begin{cases} \mathcal{Y}_3(\mathcal{Y}_2(\mathcal{Y}_1(w_{(a_{i_1})}, x_0)w_{(a_{i_2})}, y_2)w_{(a_{i_3})}, x_3)w_{(a_{i_4})}, & i_1 = 1, i_2 = 2, i_3 = 3, i_4 = 4, \\ 0, & \text{otherwise} \end{cases}
\end{aligned} \tag{4.16}
$$

for $a_1, \ldots, a_7, a_{i_1}, \cdots, a_{i_4} \in \mathcal{A}$, $\mathcal{Y}_1 \in \mathcal{V}_{a_1 a_2}^{a_7}$, $\mathcal{Y}_2 \in \mathcal{V}_{a_7 a_3}^{a_6}$, $\mathcal{Y}_3 \in \mathcal{V}_{a_6 a_4}^{a_5}$, and $w_{(a_{i_k})} \in W^{a_{i_k}}$ with $k = 1, \cdots, 4$. Then we have an isomorphism

$$
\widetilde{\mathbf{II}} : \frac{\coprod_{a_1, \cdots, a_7 \in \mathcal{A}} \mathcal{V}_{a_1 a_2}^{a_7} \otimes \mathcal{V}_{a_7 a_3}^{a_6} \otimes \mathcal{V}_{a_6 a_4}^{a_5}}{\text{Ker } \mathbf{II}} \longrightarrow \mathbf{II}\left(\coprod_{a_1, \cdots, a_7 \in \mathcal{A}} \mathcal{V}_{a_1 a_2}^{a_7} \otimes \mathcal{V}_{a_7 a_3}^{a_6} \otimes \mathcal{V}_{a_6 a_4}^{a_5}\right). \tag{4.17}
$$

The fifth one is

$$
\mathbf{PI} : \coprod_{a_1, \cdots, a_7 \in \mathcal{A}} \mathcal{V}_{a_1 a_6}^{a_5} \otimes \mathcal{V}_{a_2 a_3}^{a_7} \otimes \mathcal{V}_{a_7 a_4}^{a_6} \longrightarrow (\text{Hom}(W \otimes W \otimes W \otimes W, W))\{x_1, y_2, x_3\} \tag{4.18}
$$

defined using products and iterates of intertwining operators as follows: For

$$
\mathcal{Z} \in \coprod_{a_1, \cdots, a_7 \in \mathcal{A}} \mathcal{V}_{a_1 a_6}^{a_5} \otimes \mathcal{V}_{a_2 a_3}^{a_7} \otimes \mathcal{V}_{a_7 a_4}^{a_6}, \tag{4.19}
$$

the element $\mathbf{PI}(\mathcal{Z})$ to be defined can also be viewed as a map from $W\otimes W\otimes W\otimes W$ to $W\{x_1,y_2,x_3\}$. Then we define $\mathbf{PI}$ by linearity and by

$$
\begin{aligned}
&(\mathbf{PI}(\mathcal{Y}_1\otimes\mathcal{Y}_2\otimes\mathcal{Y}_3))(w_{(a_{i_1})},w_{(a_{i_2})},w_{(a_{i_3})},w_{(a_{i_4})};x_1,y_2,x_3)\\
&\quad=\begin{cases}\mathcal{Y}_1(w_{(a_{i_1})},x_1)\mathcal{Y}_3(\mathcal{Y}_2(w_{(a_{i_2})},y_2)w_{(a_{i_3})},x_3)w_{(a_{i_4})}, & i_1=1,i_2=2,i_3=3,i_4=4,\\ 0, & \text{otherwise}\end{cases}
\end{aligned}\tag{4.20}
$$

for $a_1,\dots,a_7,a_{i_1},\cdots,a_{i_4}\in\mathcal{A}$, $\mathcal{Y}_1\in\mathcal{V}_{a_1a_6}^{a_5}$, $\mathcal{Y}_2\in\mathcal{V}_{a_2a_3}^{a_7}$, $\mathcal{Y}_3\in\mathcal{V}_{a_7a_4}^{a_6}$, and $w_{(a_{i_k})}\in W^{a_{i_k}}$ with $k=1,\cdots,4$. Then we have an isomorphism

$$
\widetilde{\mathbf{PI}}:\frac{\coprod_{a_1,\cdots,a_7\in\mathcal{A}}\mathcal{V}_{a_1a_6}^{a_5}\otimes\mathcal{V}_{a_2a_3}^{a_7}\otimes\mathcal{V}_{a_7a_4}^{a_6}}{\mathrm{Ker}\ \mathbf{PI}}\longrightarrow\mathbf{PI}\left(\coprod_{a_1,\cdots,a_7\in\mathcal{A}}\mathcal{V}_{a_1a_6}^{a_5}\otimes\mathcal{V}_{a_2a_3}^{a_7}\otimes\mathcal{V}_{a_7a_4}^{a_6}\right).\tag{4.21}
$$

From the above linear maps, we can define a linear map:

$$
\mathcal{F}_{12}^{(1)}:\ \frac{\coprod_{a_1,\cdots,a_7\in\mathcal{A}}\mathcal{V}_{a_1a_6}^{a_5}\otimes\mathcal{V}_{a_2a_7}^{a_6}\otimes\mathcal{V}_{a_3a_4}^{a_7}}{\mathrm{Ker}\ \mathbf{PP}}\longrightarrow\frac{\coprod_{a_1,\cdots,a_7\in\mathcal{A}}\mathcal{V}_{a_1a_2}^{a_6}\otimes\mathcal{V}_{a_6a_7}^{a_5}\otimes\mathcal{V}_{a_3a_4}^{a_7}}{\mathrm{Ker}\ \mathbf{IP}}\tag{4.22}
$$

defined by linearity and by

$$
\mathcal{F}_{12}^{(1)}(\mathcal{Y}_1\otimes\mathcal{Y}_2\otimes\mathcal{Y}_3+\mathrm{Ker}\ \mathbf{PP})=\left(\sum_{a\in\mathcal{A}}\sum_{i=1}^{\mathcal{N}_{a_1a_2}^{a}\mathcal{N}_{aa_7}^{a_5}}\mathcal{Y}_{4,i}^{a}\otimes\mathcal{Y}_{5,i}^{a}\right)\otimes\mathcal{Y}_3+\mathrm{Ker}\ \mathbf{IP}\tag{4.23}
$$

for any $a_1,\dots,a_7\in\mathcal{A}$, $\mathcal{Y}_1\in\mathcal{V}_{a_1a_6}^{a_5}$, $\mathcal{Y}_2\in\mathcal{V}_{a_2a_7}^{a_6}$ and $\mathcal{Y}_3\in\mathcal{V}_{a_3a_4}^{a_7}$, where for such $\mathcal{Y}_1\in\mathcal{V}_{a_1a_6}^{a_5}$ and $\mathcal{Y}_2\in\mathcal{V}_{a_2a_7}^{a_6}$,

$$
\{\mathcal{Y}_{4,i}^{a}\in\mathcal{V}_{a_1a_2}^{a},\mathcal{Y}_{5,i}^{a}\in\mathcal{V}_{aa_7}^{a_5}\mid i=1,\cdots,\mathcal{N}_{a_1a_2}^{a}\mathcal{N}_{aa_7}^{a_5},a\in\mathcal{A}\}\tag{4.24}
$$

is a set of intertwining operators satisfying Conclusion 1 of Lemma 2.9.

Lemma 4.1. *The map $\mathcal{F}_{12}^{(1)}$ is well defined and is isomorphic. Moreover, we have*

$$
\begin{aligned}
&\langle w',(\widetilde{\mathbf{IP}}(\mathcal{F}_{12}^{(1)}(\mathcal{Z}+\mathrm{Ker}\ \mathbf{PP})))(w_1,w_2,w_3,w_4;x_0,x_2,x_3)\rangle_W\Big|_{\substack{x_0^n=e^{n\log(z_1-z_2)}\\ x_2^n=e^{n\log z_2},x_3^n=e^{n\log z_3}}}\\
&\quad=\langle w',(\widetilde{\mathbf{PP}}(\mathcal{Z}+\mathrm{Ker}\ \mathbf{PP}))(w_1,w_2,w_3,w_4;x_1,x_2,x_3)\rangle_W\Big|_{\substack{x_1^n=e^{n\log z_1}\\ x_2^n=e^{n\log z_2},x_3^n=e^{n\log z_3}}}
\end{aligned}\tag{4.25}
$$

for $w_1,w_2,w_3,w_4\in W$, $w'\in W'$ and $\mathcal{Z}\in\coprod_{a_1,\cdots,a_7\in\mathcal{A}}\mathcal{V}_{a_1a_6}^{a_5}\otimes\mathcal{V}_{a_2a_7}^{a_6}\otimes\mathcal{V}_{a_3a_4}^{a_7}$ on the region

$$
\begin{aligned}
R\ =\ &\{(z_1,z_2,z_3)\in\mathbb{C}^3\mid \mathrm{Re}z_1>\mathrm{Re}z_2>\mathrm{Re}z_3>0,\ \mathrm{Re}z_2>\mathrm{Re}(z_1-z_2)>0,\\
&\mathrm{Re}(z_2-z_3)>\mathrm{Re}(z_1-z_2)>0,\ \mathrm{Im}z_1>\mathrm{Im}z_2>\mathrm{Im}z_3>0,\ \mathrm{Im}z_2>\mathrm{Im}(z_1-z_2)>0,\\
&\mathrm{Im}(z_2-z_3)>\mathrm{Im}(z_1-z_2)>0\}.
\end{aligned}\tag{4.26}
$$

(Note: R is a nonempty simply connected region.)

Proof. First of all, we define a linear map:

$$\phi : \coprod_{a_1,\cdots,a_7\in\mathcal{A}} \mathcal{V}_{a_1a_6}^{a_5}\otimes\mathcal{V}_{a_2a_7}^{a_6}\otimes\mathcal{V}_{a_3a_4}^{a_7} \longrightarrow \frac{\coprod_{a_1,\cdots,a_7\in\mathcal{A}} \mathcal{V}_{a_1a_2}^{a_6}\otimes\mathcal{V}_{a_6a_7}^{a_5}\otimes\mathcal{V}_{a_3a_4}^{a_7}}{\mathrm{Ker}\ \mathbf{IP}} \tag{4.27}$$

defined by linearity and by

$$\phi(\mathcal{Y}_1\otimes\mathcal{Y}_2\otimes\mathcal{Y}_3) = \left(\sum_{a\in\mathcal{A}}\sum_{i=1}^{\mathcal{N}_{a_1a_2}^{a}\mathcal{N}_{aa_7}^{a_5}} \mathcal{Y}_{4,i}^{a}\otimes\mathcal{Y}_{5,i}^{a}\right)\otimes\mathcal{Y}_3 + \mathrm{Ker}\ \mathbf{IP} \tag{4.28}$$

for any $a_1,\ldots,a_7\in\mathcal{A}$, $\mathcal{Y}_1\in\mathcal{V}_{a_1a_6}^{a_5}$, $\mathcal{Y}_2\in\mathcal{V}_{a_2a_7}^{a_6}$ and $\mathcal{Y}_3\in\mathcal{V}_{a_3a_4}^{a_7}$, where for such $\mathcal{Y}_1\in\mathcal{V}_{a_1a_6}^{a_5}$ and $\mathcal{Y}_2\in\mathcal{V}_{a_2a_7}^{a_6}$,

$$\{\mathcal{Y}_{4,i}^{a}\in\mathcal{V}_{a_1a_2}^{a},\mathcal{Y}_{5,i}^{a}\in\mathcal{V}_{aa_7}^{a_5}\mid i=1,\cdots,\mathcal{N}_{a_1a_2}^{a}\mathcal{N}_{aa_7}^{a_5}, a\in\mathcal{A}\} \tag{4.29}$$

is a set of intertwining operators satisfying Conclusion 1 of Lemma 2.9. This is well defined, for if $\{\tilde{\mathcal{Y}}_{4,i}^{a},\tilde{\mathcal{Y}}_{5,i}^{a}\mid i=1,\cdots,\mathcal{N}_{a_1a_2}^{a}\mathcal{N}_{aa_7}^{a_5}, a\in\mathcal{A}\}$ is another set of intertwining operators satisfying Conclusion 1 of Lemma 2.9 for $\mathcal{Y}_1\in\mathcal{V}_{a_1a_6}^{a_5}$ and $\mathcal{Y}_2\in\mathcal{V}_{a_2a_7}^{a_6}$, then

$$\Big(\sum_{a\in\mathcal{A}}\sum_{i=1}^{\mathcal{N}_{a_1a_2}^{a}\mathcal{N}_{aa_7}^{a_5}}\tilde{\mathcal{Y}}_{4,i}^{a}\otimes\tilde{\mathcal{Y}}_{5,i}^{a}\Big)-\Big(\sum_{a\in\mathcal{A}}\sum_{i=1}^{\mathcal{N}_{a_1a_2}^{a}\mathcal{N}_{aa_7}^{a_5}}\mathcal{Y}_{4,i}^{a}\otimes\mathcal{Y}_{5,i}^{a}\Big)\in\Big(\coprod_{a\in\mathcal{A}}\mathcal{V}_{a_1a_2}^{a}\otimes\mathcal{V}_{aa_7}^{a_5}\Big)\bigcap\mathrm{Ker}\ \mathbf{I} \tag{4.30}$$

by Lemma 2.9, and one verifies at once that any element in $(\coprod_{a\in\mathcal{A}}\mathcal{V}_{a_1a_2}^{a}\otimes\mathcal{V}_{aa_7}^{a_5})\cap\mathrm{Ker}\ \mathbf{I}$ tensoring with $\mathcal{Y}_3$ lies in Ker $\mathbf{IP}$. So to prove the first statement of this lemma, it suffices to prove that ϕ is surjective and Ker $\phi=$ Ker $\mathbf{PP}$. In analogy with the proof of Proposition 2.11, it can be easily verified that ϕ is surjective. To demonstrate that Ker $\phi=$ Ker $\mathbf{PP}$, we shall digress to prove that

$$\begin{aligned}&\langle w',(\mathbf{PP}(\mathcal{Z}))(w_1,w_2,w_3,w_4;x_1,x_2,x_3)\rangle_W\Big|_{\substack{x_1^n=e^{n\log z_1}\\ x_2^n=e^{n\log z_2},x_3^n=e^{n\log z_3}}}\\ &\quad=\langle w',(\widetilde{\mathbf{IP}}(\phi(\mathcal{Z})))(w_1,w_2,w_3,w_4;x_0,x_2,x_3)\rangle_W\Big|_{\substack{x_0^n=e^{n\log(z_1-z_2)}\\ x_2^n=e^{n\log z_2},x_3^n=e^{n\log z_3}}}\end{aligned} \tag{4.31}$$

for any $w_1,w_2,w_3,w_4\in W$, $w'\in W'$ and $\mathcal{Z}\in\coprod_{a_1,\cdots,a_7\in\mathcal{A}}\mathcal{V}_{a_1a_6}^{a_5}\otimes\mathcal{V}_{a_2a_7}^{a_6}\otimes\mathcal{V}_{a_3a_4}^{a_7}$ on the region R (cf. (4.26)).

Consider any $a_1,\ldots,a_7\in\mathcal{A}$, $\mathcal{Y}_1\in\mathcal{V}_{a_1a_6}^{a_5}$, $\mathcal{Y}_2\in\mathcal{V}_{a_2a_7}^{a_6}$ and $\mathcal{Y}_3\in\mathcal{V}_{a_3a_4}^{a_7}$. Moreover, for such $\mathcal{Y}_1\in\mathcal{V}_{a_1a_6}^{a_5}$ and $\mathcal{Y}_2\in\mathcal{V}_{a_2a_7}^{a_6}$, we choose intertwining operators $\mathcal{Y}_{4,i}^{a}\in\mathcal{V}_{a_1a_2}^{a}$, $\mathcal{Y}_{5,i}^{a}\in\mathcal{V}_{aa_7}^{a_5}$ for $i=1,\cdots,\mathcal{N}_{a_1a_2}^{a}\mathcal{N}_{aa_7}^{a_5}$, $a\in\mathcal{A}$, such that they satisfy Conclusion 1 of Lemma 2.9. Then we have

$$\phi(\mathcal{Y}_1\otimes\mathcal{Y}_2\otimes\mathcal{Y}_3) = \left(\sum_{a\in\mathcal{A}}\sum_{i=1}^{\mathcal{N}_{a_1a_2}^{a}\mathcal{N}_{aa_7}^{a_5}} \mathcal{Y}_{4,i}^{a}\otimes\mathcal{Y}_{5,i}^{a}\right)\otimes\mathcal{Y}_3 + \mathrm{Ker}\ \mathbf{IP}. \tag{4.32}$$

By Lemma 2.9, for any $w_{(a_k)} \in W^{a_k}$, $k = 1, \cdots, 4$, $w'_{(a_5)} \in (W^{a_5})'$ and $r \in \mathbb{C}$, we have

$$\begin{aligned}
&\langle w'_{(a_5)}, \mathcal{Y}_1(w_{(a_1)}, x_1)\mathcal{Y}_2(w_{(a_2)}, x_2)((\mathcal{Y}_3)_{(r)}(w_{(a_3)})w_{(a_4)})\rangle_{W^{a_5}}|_{x_1^n=e^{n\log z_1}, x_2^n=e^{n\log z_2}} \\
&= \sum_{a\in\mathcal{A}} \sum_{i=1}^{\mathcal{N}_{a_1a_2}^{a}\mathcal{N}_{aa_7}^{a_5}} \langle w'_{(a_5)}, \mathcal{Y}_{5,i}^{a}(\mathcal{Y}_{4,i}^{a}(w_{(a_1)}, x_0)w_{(a_2)}, x_2) \\
&\qquad ((\mathcal{Y}_3)_{(r)}(w_{(a_3)})w_{(a_4)})\rangle_{W^{a_5}}|_{x_0^n=e^{n\log(z_1-z_2)}, x_2^n=e^{n\log z_2}}
\end{aligned} \tag{4.33}$$

on the region $\{(z_1, z_2) \in \mathbb{C}^2 \mid \mathrm{Re}z_1 > \mathrm{Re}z_2 > \mathrm{Re}(z_1 - z_2) > 0,\ \mathrm{Im}z_1 > \mathrm{Im}z_2 > \mathrm{Im}(z_1 - z_2) > 0\}$, where

$$(\mathcal{Y}_3)_{(r)}(w_{(a_3)})w_{(a_4)} = \mathrm{Res}_{x_3} x_3^r \mathcal{Y}_3(w_{(a_3)}, x_3)w_{(a_4)} \tag{4.34}$$

and

$$\mathcal{Y}_3(w_{(a_3)}, x_3)w_{(a_4)} = \sum_{r\in\mathbb{C}} (\mathcal{Y}_3)_{(r)}(w_{(a_3)})w_{(a_4)}x_3^{-r-1}. \tag{4.35}$$

So for any $w_{(a_k)} \in W^{a_k}$, $k = 1, \cdots, 4$, $w'_{(a_5)} \in (W^{a_5})'$, we can further obtain that

$$\begin{aligned}
&\langle w'_{(a_5)}, \mathcal{Y}_1(w_{(a_1)}, x_1)\mathcal{Y}_2(w_{(a_2)}, x_2)\mathcal{Y}_3(w_{(a_3)}, x_3)w_{(a_4)}\rangle_{W^{a_5}}|_{x_1^n=e^{n\log z_1}, x_2^n=e^{n\log z_2}, x_3^n=e^{n\log z_3}} \\
&= \sum_{a\in\mathcal{A}} \sum_{i=1}^{\mathcal{N}_{a_1a_2}^{a}\mathcal{N}_{aa_7}^{a_5}} \langle w'_{(a_5)}, \mathcal{Y}_{5,i}^{a}(\mathcal{Y}_{4,i}^{a}(w_{(a_1)}, x_0)w_{(a_2)}, x_2) \\
&\qquad \mathcal{Y}_3(w_{(a_3)}, x_3)w_{(a_4)}\rangle_{W^{a_5}}|_{x_0^n=e^{n\log(z_1-z_2)}, x_2^n=e^{n\log z_2}, x_3^n=e^{n\log z_3}}
\end{aligned} \tag{4.36}$$

on the region R. By the definition of **PP** and **IP**, we can derive that for any $w_1, w_2, w_3, w_4 \in W$ and $w' \in W'$,

$$\begin{aligned}
&\langle w', (\mathbf{PP}(\mathcal{Y}_1 \otimes \mathcal{Y}_2 \otimes \mathcal{Y}_3))(w_1, w_2, w_3, w_4; x_1, x_2, x_3)\rangle_W|_{x_1^n=e^{n\log z_1}, x_2^n=e^{n\log z_2}, x_3^n=e^{n\log z_3}} \\
&= \langle w', (\mathbf{IP}(\sum_{a\in\mathcal{A}} \sum_{i=1}^{\mathcal{N}_{a_1a_2}^{a}\mathcal{N}_{aa_7}^{a_5}} \mathcal{Y}_{4,i}^{a} \otimes \mathcal{Y}_{5,i}^{a} \otimes \mathcal{Y}_3)) \\
&\qquad (w_1, w_2, w_3, w_4; x_0, x_2, x_3)\rangle_W|_{x_0^n=e^{n\log(z_1-z_2)}, x_2^n=e^{n\log z_2}, x_3^n=e^{n\log z_3}} \\
&= \langle w', (\widetilde{\mathbf{IP}}(\sum_{a\in\mathcal{A}} \sum_{i=1}^{\mathcal{N}_{a_1a_2}^{a}\mathcal{N}_{aa_7}^{a_5}} \mathcal{Y}_{4,i}^{a} \otimes \mathcal{Y}_{5,i}^{a} \otimes \mathcal{Y}_3 + \mathrm{Ker}\,\mathbf{IP})) \\
&\qquad (w_1, w_2, w_3, w_4; x_0, x_2, x_3)\rangle_W|_{x_0^n=e^{n\log(z_1-z_2)}, x_2^n=e^{n\log z_2}, x_3^n=e^{n\log z_3}} \\
&= \langle w', (\widetilde{\mathbf{IP}}(\phi(\mathcal{Y}_1 \otimes \mathcal{Y}_2 \otimes \mathcal{Y}_3))) \\
&\qquad (w_1, w_2, w_3, w_4; x_0, x_2, x_3)\rangle_W|_{x_0^n=e^{n\log(z_1-z_2)}, x_2^n=e^{n\log z_2}, x_3^n=e^{n\log z_3}}
\end{aligned} \tag{4.37}$$

on the region R. Therefore, (4.31) holds by linearity.

Now we can prove that Ker ϕ = Ker **PP**. For any $\mathcal{Z} \in \mathrm{Ker}\,\phi$, we have $\phi(\mathcal{Z}) = \mathrm{Ker}\,\mathbf{IP}$. Then by (4.31) we obtain

$$\langle w', (\mathbf{PP}(\mathcal{Z}))(w_1, w_2, w_3, w_4; x_1, x_2, x_3)\rangle_W|_{x_1^n=e^{n\log z_1}, x_2^n=e^{n\log z_2}, x_3^n=e^{n\log z_3}} = 0 \tag{4.38}$$

for any $w_1, w_2, w_3, w_4 \in W$ and $w' \in W'$ on the region R. Note that the left-hand side of (4.38) is analytic in z_1, z_2 and z_3 for z_1, z_2, z_3 satisfying $|z_1| > |z_2| > |z_3| > 0$. Also note that the region R is a subset of the domain $|z_1| > |z_2| > |z_3| > 0$ of this analytic function. From the basic properties of analytic functions, (4.38) implies that the left-hand side of (4.38) as an analytic function is 0 for all z_1, z_2 and z_3 satisfying $|z_1| > |z_2| > |z_3| > 0$. Thus for any $w_1, w_2, w_3, w_4 \in W$, $w' \in W'$,

$$\langle w', (\mathbf{PP}(\mathcal{Z}))(w_1, w_2, w_3, w_4; x_1, x_2, x_3)\rangle_W = 0. \tag{4.39}$$

By the definition of the map $\mathbf{PP}$, we therefore have $\mathbf{PP}(\mathcal{Z}) = 0$; namely, $\mathcal{Z} \in \mathrm{Ker}\ \mathbf{PP}$. So $\mathrm{Ker}\ \phi \subseteq \mathrm{Ker}\ \mathbf{PP}$. On the other hand, for any $\mathcal{Z} \in \mathrm{Ker}\ \mathbf{PP}$, we have

$$\langle w', (\widetilde{\mathbf{IP}}(\phi(\mathcal{Z})))(w_1, w_2, w_3, w_4; x_0, x_2, x_3)\rangle_W\Big|_{x_0^n = e^{n\log(z_1 - z_2)}, x_2^n = e^{n \log z_2}, x_3^n = e^{n\log z_3}} = 0 \tag{4.40}$$

on R by (4.31). In analogy with the above discussion for (4.38), we can similarly prove that $\widetilde{\mathbf{IP}}(\phi(\mathcal{Z})) = 0$. Since $\widetilde{\mathbf{IP}}$ is isomorphic, we deduce that $\phi(\mathcal{Z}) = \mathrm{Ker}\ \mathbf{IP}$, which further implies that $\mathcal{Z} \in \mathrm{Ker}\ \phi$. So $\mathrm{Ker}\ \mathbf{PP} \subseteq \mathrm{Ker}\ \phi$. To sum up, we deduce that $\mathrm{Ker}\ \phi = \mathrm{Ker}\ \mathbf{PP}$.

So by the definition of ϕ and the fact that $\mathrm{Ker}\ \phi = \mathrm{Ker}\ \mathbf{PP}$, we have an isomorphism $\tilde{\phi}$ which makes the following diagram commute:

$$\begin{tikzcd}
\coprod\limits_{a_1,\cdots,a_7\in\mathcal{A}} \mathcal{V}_{a_1a_6}^{a_5}\otimes\mathcal{V}_{a_2a_7}^{a_6}\otimes\mathcal{V}_{a_3a_4}^{a_7} \arrow[r, "\phi"] \arrow[d, "\pi"'] & \dfrac{\coprod\limits_{a_1,\cdots,a_7\in\mathcal{A}} \mathcal{V}_{a_1a_2}^{a_6}\otimes\mathcal{V}_{a_6a_7}^{a_5}\otimes\mathcal{V}_{a_3a_4}^{a_7}}{\mathrm{Ker}\ \mathbf{IP}}. \\
\dfrac{\coprod\limits_{a_1,\cdots,a_7\in\mathcal{A}} \mathcal{V}_{a_1a_6}^{a_5}\otimes\mathcal{V}_{a_2a_7}^{a_6}\otimes\mathcal{V}_{a_3a_4}^{a_7}}{\mathrm{Ker}\ \mathbf{PP}} \arrow[ur, "\tilde{\phi}"] &
\end{tikzcd} \tag{4.41}$$

Observe that $\mathcal{F}_{12}^{(1)}$ coincides with $\tilde{\phi}$. So $\mathcal{F}_{12}^{(1)}$ is well defined and is isomorphic. Moreover, by (4.31) we get

$$\begin{aligned}
&\langle w', (\widetilde{\mathbf{IP}}(\mathcal{F}_{12}^{(1)}(\mathcal{Z} + \mathrm{Ker}\ \mathbf{PP})))(w_1, w_2, w_3, w_4; x_0, x_2, x_3)\rangle_W\Big|_{\substack{x_0^n = e^{n\log(z_1-z_2)}\\ x_2^n = e^{n\log z_2}, x_3^n = e^{n\log z_3}}} \\
&= \langle w', (\widetilde{\mathbf{IP}}(\phi(\mathcal{Z})))(w_1, w_2, w_3, w_4; x_0, x_2, x_3)\rangle_W\Big|_{\substack{x_0^n = e^{n\log(z_1-z_2)}\\ x_2^n = e^{n\log z_2}, x_3^n = e^{n\log z_3}}} \\
&= \langle w', (\mathbf{PP}(\mathcal{Z}))(w_1, w_2, w_3, w_4; x_1, x_2, x_3)\rangle_W\Big|_{\substack{x_1^n = e^{n\log z_1}\\ x_2^n = e^{n\log z_2}, x_3^n = e^{n\log z_3}}} \\
&= \langle w', (\widetilde{\mathbf{PP}}(\mathcal{Z} + \mathrm{Ker}\ \mathbf{PP}))(w_1, w_2, w_3, w_4; x_1, x_2, x_3)\rangle_W\Big|_{\substack{x_1^n = e^{n\log z_1}\\ x_2^n = e^{n\log z_2}, x_3^n = e^{n\log z_3}}}
\end{aligned} \tag{4.42}$$

for $w_1, w_2, w_3, w_4 \in W$, $w' \in W'$ and $\mathcal{Z} \in \coprod_{a_1,\cdots,a_7\in\mathcal{A}} \mathcal{V}_{a_1a_6}^{a_5} \otimes \mathcal{V}_{a_2a_7}^{a_6} \otimes \mathcal{V}_{a_3a_4}^{a_7}$ on the region R. Namely, (4.25) holds. Therefore, this lemma holds. ∎

In analogy with Proposition 4.1, we have another four isomorphisms. The first one is:

$$\mathcal{F}_{23}^{(1)} : \frac{\coprod_{a_1,\cdots,a_7\in\mathcal{A}} \mathcal{V}_{a_1a_6}^{a_5} \otimes \mathcal{V}_{a_2a_7}^{a_6} \otimes \mathcal{V}_{a_3a_4}^{a_7}}{\text{Ker } \mathbf{PP}} \longrightarrow \frac{\coprod_{a_1,\cdots,a_7\in\mathcal{A}} \mathcal{V}_{a_1a_6}^{a_5} \otimes \mathcal{V}_{a_2a_3}^{a_7} \otimes \mathcal{V}_{a_7a_4}^{a_6}}{\text{Ker } \mathbf{PI}} \tag{4.43}$$

defined by linearity and by

$$\mathcal{F}_{23}^{(1)}(\mathcal{Y}_1 \otimes \mathcal{Y}_2 \otimes \mathcal{Y}_3 + \text{Ker } \mathbf{PP}) = \sum_{a\in\mathcal{A}} \sum_{i=1}^{\mathcal{N}_{a_2a_3}^{a}\mathcal{N}_{aa_4}^{a_6}} \mathcal{Y}_1 \otimes \mathcal{Y}_{4,i}^{a} \otimes \mathcal{Y}_{5,i}^{a} + \text{Ker } \mathbf{PI} \tag{4.44}$$

for any $a_1, \ldots, a_7 \in \mathcal{A}$, $\mathcal{Y}_1 \in \mathcal{V}_{a_1a_6}^{a_5}$, $\mathcal{Y}_2 \in \mathcal{V}_{a_2a_7}^{a_6}$ and $\mathcal{Y}_3 \in \mathcal{V}_{a_3a_4}^{a_7}$, where for such $\mathcal{Y}_2 \in \mathcal{V}_{a_2a_7}^{a_6}$ and $\mathcal{Y}_3 \in \mathcal{V}_{a_3a_4}^{a_7}$,

$$\{\mathcal{Y}_{4,i}^{a} \in \mathcal{V}_{a_2a_3}^{a}, \mathcal{Y}_{5,i}^{a} \in \mathcal{V}_{aa_4}^{a_6} \mid i = 1, \cdots, \mathcal{N}_{a_2a_3}^{a}\mathcal{N}_{aa_4}^{a_6}, a \in \mathcal{A}\}$$

is a set of intertwining operators satisfying Conclusion 1 of Lemma 2.9. The second one is:

$$\mathcal{F}_{13} : \frac{\coprod_{a_1,\cdots,a_7\in\mathcal{A}} \mathcal{V}_{a_1a_6}^{a_5} \otimes \mathcal{V}_{a_2a_3}^{a_7} \otimes \mathcal{V}_{a_7a_4}^{a_6}}{\text{Ker } \mathbf{PI}} \longrightarrow \frac{\coprod_{a_1,\cdots,a_7\in\mathcal{A}} \mathcal{V}_{a_1a_7}^{a_6} \otimes \mathcal{V}_{a_2a_3}^{a_7} \otimes \mathcal{V}_{a_6a_4}^{a_5}}{\text{Ker } \mathbf{I}^{\mathbf{P}}} \tag{4.45}$$

defined by linearity and by

$$\mathcal{F}_{13}(\mathcal{Y}_1 \otimes \mathcal{Y}_2 \otimes \mathcal{Y}_3 + \text{Ker } \mathbf{PI}) = \sum_{a\in\mathcal{A}} \sum_{i=1}^{\mathcal{N}_{a_1a_7}^{a}\mathcal{N}_{aa_4}^{a_5}} \mathcal{Y}_{4,i}^{a} \otimes \mathcal{Y}_2 \otimes \mathcal{Y}_{5,i}^{a} + \text{Ker } \mathbf{I}^{\mathbf{P}} \tag{4.46}$$

for any $a_1, \ldots, a_7 \in \mathcal{A}$, $\mathcal{Y}_1 \in \mathcal{V}_{a_1a_6}^{a_5}$, $\mathcal{Y}_2 \in \mathcal{V}_{a_2a_3}^{a_7}$ and $\mathcal{Y}_3 \in \mathcal{V}_{a_7a_4}^{a_6}$, where for such $\mathcal{Y}_1 \in \mathcal{V}_{a_1a_6}^{a_5}$ and $\mathcal{Y}_3 \in \mathcal{V}_{a_7a_4}^{a_6}$,

$$\{\mathcal{Y}_{4,i}^{a} \in \mathcal{V}_{a_1a_7}^{a}, \mathcal{Y}_{5,i}^{a} \in \mathcal{V}_{aa_4}^{a_5} \mid i = 1, \cdots, \mathcal{N}_{a_1a_7}^{a}\mathcal{N}_{aa_4}^{a_5}, a \in \mathcal{A}\}$$

is a set of intertwining operators satisfying Conclusion 1 of Lemma 2.9. The third one is:

$$\mathcal{F}_{12}^{(2)} : \frac{\coprod_{a_1,\cdots,a_7\in\mathcal{A}} \mathcal{V}_{a_1a_7}^{a_6} \otimes \mathcal{V}_{a_2a_3}^{a_7} \otimes \mathcal{V}_{a_6a_4}^{a_5}}{\text{Ker } \mathbf{I}^{\mathbf{P}}} \longrightarrow \frac{\coprod_{a_1,\cdots,a_7\in\mathcal{A}} \mathcal{V}_{a_1a_2}^{a_7} \otimes \mathcal{V}_{a_7a_3}^{a_6} \otimes \mathcal{V}_{a_6a_4}^{a_5}}{\text{Ker } \mathbf{II}} \tag{4.47}$$

defined by linearity and by

$$\mathcal{F}_{12}^{(2)}(\mathcal{Y}_1 \otimes \mathcal{Y}_2 \otimes \mathcal{Y}_3 + \text{Ker } \mathbf{I}^{\mathbf{P}}) = \sum_{a\in\mathcal{A}} \sum_{i=1}^{\mathcal{N}_{a_1a_2}^{a}\mathcal{N}_{aa_3}^{a_6}} \mathcal{Y}_{4,i}^{a} \otimes \mathcal{Y}_{5,i}^{a} \otimes \mathcal{Y}_3 + \text{Ker } \mathbf{II} \tag{4.48}$$

for any $a_1, \ldots, a_7 \in \mathcal{A}$, $\mathcal{Y}_1 \in \mathcal{V}_{a_1a_7}^{a_6}$, $\mathcal{Y}_2 \in \mathcal{V}_{a_2a_3}^{a_7}$ and $\mathcal{Y}_3 \in \mathcal{V}_{a_6a_4}^{a_5}$, where for such $\mathcal{Y}_1 \in \mathcal{V}_{a_1a_7}^{a_6}$ and $\mathcal{Y}_2 \in \mathcal{V}_{a_2a_3}^{a_7}$,

$$\{\mathcal{Y}_{4,i}^a \in \mathcal{V}_{a_1a_2}^a, \mathcal{Y}_{5,i}^a \in \mathcal{V}_{aa_3}^{a_6} \mid i = 1, \cdots, \mathcal{N}_{a_1a_2}^a \mathcal{N}_{aa_3}^{a_6}, a \in \mathcal{A}\}$$

is a set of intertwining operators satisfying Conclusion 1 of Lemma 2.9. The fourth one is:

$$\mathcal{F}_{23}^{(2)} : \frac{\coprod_{a_1,\cdots,a_7 \in \mathcal{A}} \mathcal{V}_{a_1a_2}^{a_6} \otimes \mathcal{V}_{a_6a_7}^{a_5} \otimes \mathcal{V}_{a_3a_4}^{a_7}}{\text{Ker } \mathbf{IP}} \longrightarrow \frac{\coprod_{a_1,\cdots,a_7 \in \mathcal{A}} \mathcal{V}_{a_1a_2}^{a_6} \otimes \mathcal{V}_{a_6a_3}^{a_7} \otimes \mathcal{V}_{a_7a_4}^{a_5}}{\text{Ker } \mathbf{II}} \tag{4.49}$$

defined by linearity and by

$$\mathcal{F}_{23}^{(2)}(\mathcal{Y}_1 \otimes \mathcal{Y}_2 \otimes \mathcal{Y}_3 + \text{Ker } \mathbf{IP}) = \sum_{a \in \mathcal{A}} \sum_{i=1}^{\mathcal{N}_{a_6a_3}^a \mathcal{N}_{aa_4}^{a_5}} \mathcal{Y}_1 \otimes \mathcal{Y}_{4,i}^a \otimes \mathcal{Y}_{5,i}^a + \text{Ker } \mathbf{II} \tag{4.50}$$

for any $a_1, \ldots, a_7 \in \mathcal{A}$, $\mathcal{Y}_1 \in \mathcal{V}_{a_1a_2}^{a_6}$, $\mathcal{Y}_2 \in \mathcal{V}_{a_6a_7}^{a_5}$ and $\mathcal{Y}_3 \in \mathcal{V}_{a_3a_4}^{a_7}$, where for such $\mathcal{Y}_2 \in \mathcal{V}_{a_6a_7}^{a_5}$ and $\mathcal{Y}_3 \in \mathcal{V}_{a_3a_4}^{a_7}$,

$$\{\mathcal{Y}_{4,i}^a \in \mathcal{V}_{a_6a_3}^a, \mathcal{Y}_{5,i}^a \in \mathcal{V}_{aa_4}^{a_5} \mid i = 1, \cdots, \mathcal{N}_{a_6a_3}^a \mathcal{N}_{aa_4}^{a_5}, a \in \mathcal{A}\}$$

is a set of intertwining operators satisfying Conclusion 1 of Lemma 2.9.

Also, in analogy with Lemma 4.1, we have

$$\begin{aligned}
&\langle w', (\widetilde{\mathbf{PI}}(\mathcal{F}_{23}^{(1)}(\mathcal{Z} + \text{Ker } \mathbf{PP})))(w_1, w_2, w_3, w_4; x_1, y_2, x_3)\rangle_W \Big|_{\substack{x_1^n = e^{n \log z_1}, y_2^n = e^{n \log(z_2 - z_3)} \\ x_3^n = e^{n \log z_3}}} \\
&\quad = \langle w', (\widetilde{\mathbf{PP}}(\mathcal{Z} + \text{Ker } \mathbf{PP}))(w_1, w_2, w_3, w_4; x_1, x_2, x_3)\rangle_W \Big|_{\substack{x_1^n = e^{n \log z_1}, x_2^n = e^{n \log z_2} \\ x_3^n = e^{n \log z_3}}}
\end{aligned} \tag{4.51}$$

for $w_1, w_2, w_3, w_4 \in W$, $w' \in W'$ and $\mathcal{Z} \in \coprod_{a_1,\cdots,a_7 \in \mathcal{A}} \mathcal{V}_{a_1a_6}^{a_5} \otimes \mathcal{V}_{a_2a_7}^{a_6} \otimes \mathcal{V}_{a_3a_4}^{a_7}$ on the region $\{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid \text{Re} z_1 > \text{Re} z_2 > \text{Re} z_3 > \text{Re}(z_2 - z_3) > 0,\ \text{Re}(z_1 - z_3) > \text{Re}(z_2 - z_3) > 0,\ \text{Im} z_1 > \text{Im} z_2 > \text{Im} z_3 > \text{Im}(z_2 - z_3) > 0,\ \text{Im}(z_1 - z_3) > \text{Im}(z_2 - z_3) > 0\}$;

$$\begin{aligned}
&\langle w', (\widetilde{\mathbf{I}^{\mathbf{P}}}(\mathcal{F}_{13}(\mathcal{Z} + \text{Ker } \mathbf{PI})))(w_1, w_2, w_3, w_4; y_1, y_2, x_3)\rangle_W \Big|_{\substack{y_1^n = e^{n \log(z_1 - z_3)}, y_2^n = e^{n \log(z_2 - z_3)} \\ x_3^n = e^{n \log z_3}}} \\
&\quad = \langle w', (\widetilde{\mathbf{PI}}(\mathcal{Z} + \text{Ker } \mathbf{PI}))(w_1, w_2, w_3, w_4; x_1, y_2, x_3)\rangle_W \Big|_{\substack{x_1^n = e^{n \log z_1}, y_2^n = e^{n \log(z_2 - z_3)} \\ x_3^n = e^{n \log z_3}}}
\end{aligned} \tag{4.52}$$

for $w_1, w_2, w_3, w_4 \in W$, $w' \in W'$ and $\mathcal{Z} \in \coprod_{a_1,\cdots,a_7 \in \mathcal{A}} \mathcal{V}_{a_1a_6}^{a_5} \otimes \mathcal{V}_{a_2a_3}^{a_7} \otimes \mathcal{V}_{a_7a_4}^{a_6}$ on the region $\{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid \text{Re} z_1 > \text{Re} z_3 > \text{Re}(z_1 - z_3) > \text{Re}(z_2 - z_3) > 0,\ \text{Im} z_1 > \text{Im} z_3 > \text{Im}(z_1 - z_3) > \text{Im}(z_2 - z_3) > 0\}$;

$$\begin{aligned}
&\langle w', (\widetilde{\mathbf{II}}(\mathcal{F}_{12}^{(2)}(\mathcal{Z} + \text{Ker } \mathbf{I}^{\mathbf{P}})))(w_1, w_2, w_3, w_4; x_0, y_2, x_3)\rangle_W \Big|_{\substack{x_0^n = e^{n \log(z_1 - z_2)}, y_2^n = e^{n \log(z_2 - z_3)} \\ x_3^n = e^{n \log z_3}}} \\
&\quad = \langle w', (\widetilde{\mathbf{I}^{\mathbf{P}}}(\mathcal{Z} + \text{Ker } \mathbf{I}^{\mathbf{P}}))(w_1, w_2, w_3, w_4; y_1, y_2, x_3)\rangle_W \Big|_{\substack{y_1^n = e^{n \log(z_1 - z_3)}, y_2^n = e^{n \log(z_2 - z_3)} \\ x_3^n = e^{n \log z_3}}}
\end{aligned} \tag{4.53}$$

for $w_1, w_2, w_3, w_4 \in W$, $w' \in W'$ and $\mathcal{Z} \in \coprod_{a_1,\cdots,a_7\in\mathcal{A}} \mathcal{V}_{a_1a_7}^{a_6} \otimes \mathcal{V}_{a_2a_3}^{a_7} \otimes \mathcal{V}_{a_6a_4}^{a_5}$ on the region $\{(z_1,z_2,z_3)\in\mathbb{C}^3 \mid \mathrm{Re}z_3 > \mathrm{Re}(z_1-z_3) > \mathrm{Re}(z_2-z_3) > \mathrm{Re}(z_1-z_2) > 0,\ \mathrm{Im}z_3 > \mathrm{Im}(z_1-z_3) > \mathrm{Im}(z_2-z_3) > \mathrm{Im}(z_1-z_3) > 0\}$;

$$\begin{aligned}
&\langle w', (\widetilde{\mathbf{II}}(\mathcal{F}_{23}^{(2)}(\mathcal{Z}+\mathrm{Ker}\,\mathbf{IP})))(w_1,w_2,w_3,w_4;x_0,y_2,x_3)\rangle_W\Big|_{\substack{x_0^n=e^{n\log(z_1-z_2)},y_2^n=e^{n\log(z_2-z_3)}\\ x_3^n=e^{n\log z_3}}}\\
&\quad= \langle w', (\widetilde{\mathbf{IP}}(\mathcal{Z}+\mathrm{Ker}\,\mathbf{IP}))(w_1,w_2,w_3,w_4;x_0,x_2,x_3)\rangle_W\Big|_{\substack{x_0^n=e^{n\log(z_1-z_2)},x_2^n=e^{n\log z_2}\\ x_3^n=e^{n\log z_3}}} \qquad (4.54)
\end{aligned}$$

for $w_1, w_2, w_3, w_4 \in W$, $w' \in W'$ and $\mathcal{Z} \in \coprod_{a_1,\cdots,a_7\in\mathcal{A}} \mathcal{V}_{a_1a_2}^{a_6} \otimes \mathcal{V}_{a_6a_7}^{a_5} \otimes \mathcal{V}_{a_3a_4}^{a_7}$ on the region $\{(z_1,z_2,z_3)\in\mathbb{C}^3 \mid \mathrm{Re}z_2 > \mathrm{Re}z_3 > \mathrm{Re}(z_2-z_3) > \mathrm{Re}(z_1-z_2) > 0,\ \mathrm{Re}z_1 > \mathrm{Re}z_3 > \mathrm{Re}(z_1-z_3) > 0,\ \mathrm{Im}z_2 > \mathrm{Im}z_3 > \mathrm{Im}(z_2-z_3) > \mathrm{Im}(z_1-z_2) > 0,\ \mathrm{Im}z_1 > \mathrm{Im}z_3 > \mathrm{Im}(z_1-z_3) > 0\}$.

Similarly, in analogy with the maps $\tilde{\Omega}^{(1)}$ and $(\widetilde{\Omega^{-1}})^{(1)}$ (cf. Proposition 2.12), we have the following linear maps which are well defined and are isomorphic:

$$\tilde{\Omega}^{(2)}, (\widetilde{\Omega^{-1}})^{(2)} : \frac{\coprod_{a_1,a_2,a_3,a_4,a_5\in\mathcal{A}} \mathcal{V}_{a_1a_5}^{a_4} \otimes \mathcal{V}_{a_2a_3}^{a_5}}{\mathrm{Ker}\,\mathbf{P}} \longrightarrow \frac{\coprod_{a_1,a_2,a_3,a_4,a_5\in\mathcal{A}} \mathcal{V}_{a_2a_3}^{a_5} \otimes \mathcal{V}_{a_5a_1}^{a_4}}{\mathrm{Ker}\,\mathbf{I}} \qquad (4.55)$$

determined by linearity and by

$$\tilde{\Omega}^{(2)}(\mathcal{Y}_1 \otimes \mathcal{Y}_2 + \mathrm{Ker}\,\mathbf{P}) = \mathcal{Y}_2 \otimes \Omega(\mathcal{Y}_1) + \mathrm{Ker}\,\mathbf{I}, \qquad (4.56)$$

$$(\widetilde{\Omega^{-1}})^{(2)}(\mathcal{Y}_1 \otimes \mathcal{Y}_2 + \mathrm{Ker}\,\mathbf{P}) = \mathcal{Y}_2 \otimes \Omega^{-1}(\mathcal{Y}_1) + \mathrm{Ker}\,\mathbf{I} \qquad (4.57)$$

for any $a_1, \ldots, a_5 \in \mathcal{A}$, $\mathcal{Y}_1 \in \mathcal{V}_{a_1a_5}^{a_4}$, $\mathcal{Y}_2 \in \mathcal{V}_{a_2a_3}^{a_5}$;

$$\tilde{\Omega}^{(3)}, (\widetilde{\Omega^{-1}})^{(3)} : \frac{\coprod_{a_1,a_2,a_3,a_4,a_5\in\mathcal{A}} \mathcal{V}_{a_1a_2}^{a_5} \otimes \mathcal{V}_{a_5a_3}^{a_4}}{\mathrm{Ker}\,\mathbf{I}} \longrightarrow \frac{\coprod_{a_1,a_2,a_3,a_4,a_5\in\mathcal{A}} \mathcal{V}_{a_3a_5}^{a_4} \otimes \mathcal{V}_{a_1a_2}^{a_5}}{\mathrm{Ker}\,\mathbf{P}} \qquad (4.58)$$

determined by linearity and by

$$\tilde{\Omega}^{(3)}(\mathcal{Y}_1 \otimes \mathcal{Y}_2 + \mathrm{Ker}\,\mathbf{I}) = \Omega(\mathcal{Y}_2) \otimes \mathcal{Y}_1 + \mathrm{Ker}\,\mathbf{P}, \qquad (4.59)$$

$$(\widetilde{\Omega^{-1}})^{(3)}(\mathcal{Y}_1 \otimes \mathcal{Y}_2 + \mathrm{Ker}\,\mathbf{I}) = \Omega^{-1}(\mathcal{Y}_2) \otimes \mathcal{Y}_1 + \mathrm{Ker}\,\mathbf{P} \qquad (4.60)$$

for $a_1, \ldots, a_5 \in \mathcal{A}$, $\mathcal{Y}_1 \in \mathcal{V}_{a_1a_2}^{a_5}$, $\mathcal{Y}_2 \in \mathcal{V}_{a_5a_3}^{a_4}$;

$$\tilde{\Omega}^{(4)}, (\widetilde{\Omega^{-1}})^{(4)} : \frac{\coprod_{a_1,a_2,a_3,a_4,a_5\in\mathcal{A}} \mathcal{V}_{a_1a_5}^{a_4} \otimes \mathcal{V}_{a_2a_3}^{a_5}}{\mathrm{Ker}\,\mathbf{P}} \longrightarrow \frac{\coprod_{a_1,a_2,a_3,a_4,a_5\in\mathcal{A}} \mathcal{V}_{a_1a_5}^{a_4} \otimes \mathcal{V}_{a_3a_2}^{a_5}}{\mathrm{Ker}\,\mathbf{P}} \qquad (4.61)$$

determined by linearity and by

$$\tilde{\Omega}^{(4)}(\mathcal{Y}_1 \otimes \mathcal{Y}_2 + \mathrm{Ker}\,\mathbf{P}) = \mathcal{Y}_1 \otimes \Omega(\mathcal{Y}_2) + \mathrm{Ker}\,\mathbf{P}, \qquad (4.62)$$

$$(\widetilde{\Omega^{-1}})^{(4)}(\mathcal{Y}_1 \otimes \mathcal{Y}_2 + \text{Ker } \mathbf{P}) = \mathcal{Y}_1 \otimes \Omega^{-1}(\mathcal{Y}_2) + \text{Ker } \mathbf{P} \tag{4.63}$$

for $a_1, \ldots, a_5 \in \mathcal{A}$, $\mathcal{Y}_1 \in \mathcal{V}_{a_1 a_5}^{a_4}$, $\mathcal{Y}_2 \in \mathcal{V}_{a_2 a_3}^{a_5}$. It is easy to verify that these isomorphisms have relations:

$$(\tilde{\Omega}^{(2)})^{-1} = (\widetilde{\Omega^{-1}})^{(3)}, \quad ((\widetilde{\Omega^{-1}})^{(2)})^{-1} = \tilde{\Omega}^{(3)}, \tag{4.64}$$

$$(\tilde{\Omega}^{(1)})^{-1} = (\widetilde{\Omega^{-1}})^{(1)}, \quad (\tilde{\Omega}^{(4)})^{-1} = (\widetilde{\Omega^{-1}})^{(4)}. \tag{4.65}$$

The above isomorphisms are not independent, we have the following relations of them:

Theorem 4.2. *The above isomorphisms satisfy the following genus-zero Moore-Seiberg equations:*

$$\mathcal{F}_{23}^{(2)} \circ \mathcal{F}_{12}^{(1)} = \mathcal{F}_{12}^{(2)} \circ \mathcal{F}_{13} \circ \mathcal{F}_{23}^{(1)}, \tag{4.66}$$

$$\mathcal{F} \circ \tilde{\Omega}^{(3)} \circ \mathcal{F} = \tilde{\Omega}^{(1)} \circ \mathcal{F} \circ \tilde{\Omega}^{(4)}, \tag{4.67}$$

$$\mathcal{F} \circ (\widetilde{\Omega^{-1}})^{(3)} \circ \mathcal{F} = (\widetilde{\Omega^{-1}})^{(1)} \circ \mathcal{F} \circ (\widetilde{\Omega^{-1}})^{(4)}. \tag{4.68}$$

Proof. Firstly, we shall prove eq. (4.66).

Note that $\mathcal{F}_{23}^{(2)} \circ \mathcal{F}_{12}^{(1)}$ and $\mathcal{F}_{12}^{(2)} \circ \mathcal{F}_{13} \circ \mathcal{F}_{23}^{(1)}$ are both isomorphisms from

$$\frac{\coprod_{a_1, \cdots, a_7 \in \mathcal{A}} \mathcal{V}_{a_1 a_6}^{a_5} \otimes \mathcal{V}_{a_2 a_7}^{a_6} \otimes \mathcal{V}_{a_3 a_4}^{a_7}}{\text{Ker } \mathbf{PP}} \tag{4.69}$$

to

$$\frac{\coprod_{a_1, \cdots, a_7 \in \mathcal{A}} \mathcal{V}_{a_1 a_2}^{a_6} \otimes \mathcal{V}_{a_6 a_3}^{a_7} \otimes \mathcal{V}_{a_7 a_4}^{a_5}}{\text{Ker } \mathbf{II}}. \tag{4.70}$$

By linearity, it suffices to prove that

$$\begin{aligned} &\mathcal{F}_{23}^{(2)} \circ \mathcal{F}_{12}^{(1)}(\mathcal{Y}_1 \otimes \mathcal{Y}_2 \otimes \mathcal{Y}_3 + \text{Ker } \mathbf{PP}) \\ &\quad = \mathcal{F}_{12}^{(2)} \circ \mathcal{F}_{13} \circ \mathcal{F}_{23}^{(1)}(\mathcal{Y}_1 \otimes \mathcal{Y}_2 \otimes \mathcal{Y}_3 + \text{Ker } \mathbf{PP}) \end{aligned} \tag{4.71}$$

for any $a_1, \ldots, a_7 \in \mathcal{A}$, $\mathcal{Y}_1 \in \mathcal{V}_{a_1 a_6}^{a_5}$, $\mathcal{Y}_2 \in \mathcal{V}_{a_2 a_7}^{a_6}$ and $\mathcal{Y}_3 \in \mathcal{V}_{a_3 a_4}^{a_7}$. Fix any $a_1, \ldots, a_7 \in \mathcal{A}$, $\mathcal{Y}_1 \in \mathcal{V}_{a_1 a_6}^{a_5}$, $\mathcal{Y}_2 \in \mathcal{V}_{a_2 a_7}^{a_6}$ and $\mathcal{Y}_3 \in \mathcal{V}_{a_3 a_4}^{a_7}$. Then by (4.25) and (4.51)-(4.54), we see that

$$\begin{aligned} &\langle w', (\widetilde{\mathbf{II}}(\mathcal{F}_{23}^{(2)} \circ \mathcal{F}_{12}^{(1)}(\mathcal{Y}_1 \otimes \mathcal{Y}_2 \otimes \mathcal{Y}_3 + \text{Ker } \mathbf{PP}))) \\ &\qquad (w_1, w_2, w_3, w_4; x_0, y_2, x_3)\rangle_W \big|_{x_0^n = e^{n \log(z_1 - z_2)}, y_2^n = e^{n \log(z_2 - z_3)}, x_3^n = e^{n \log z_3}} \\ &= \langle w', (\widetilde{\mathbf{IP}}(\mathcal{F}_{12}^{(1)}(\mathcal{Y}_1 \otimes \mathcal{Y}_2 \otimes \mathcal{Y}_3 + \text{Ker } \mathbf{PP}))) \\ &\qquad (w_1, w_2, w_3, w_4; x_0, x_2, x_3)\rangle_W \big|_{x_0^n = e^{n \log(z_1 - z_2)}, x_2^n = e^{n \log z_2}, x_3^n = e^{n \log z_3}} \\ &= \langle w', (\widetilde{\mathbf{PP}}(\mathcal{Y}_1 \otimes \mathcal{Y}_2 \otimes \mathcal{Y}_3 + \text{Ker } \mathbf{PP})) \\ &\qquad (w_1, w_2, w_3, w_4; x_1, x_2, x_3)\rangle_W \big|_{x_1^n = e^{n \log z_1}, x_2^n = e^{n \log z_2}, x_3^n = e^{n \log z_3}} \end{aligned} \tag{4.72}$$

and

$$
\begin{aligned}
&\langle w', (\widetilde{\mathbf{II}}(\mathcal{F}_{12}^{(2)} \circ \mathcal{F}_{13} \circ \mathcal{F}_{23}^{(1)}(\mathcal{Y}_1 \otimes \mathcal{Y}_2 \otimes \mathcal{Y}_3 + \mathrm{Ker}\ \mathbf{PP})))\\
&\qquad\qquad (w_1, w_2, w_3, w_4; x_0, y_2, x_3)\rangle_W\big|_{x_0^n = e^{n\log(z_1-z_2)}, y_2^n = e^{n\log(z_2-z_3)}, x_3^n = e^{n\log z_3}}\\
&\quad = \langle w', (\widetilde{\mathbf{I^P}}(\mathcal{F}_{13} \circ \mathcal{F}_{23}^{(1)}(\mathcal{Y}_1 \otimes \mathcal{Y}_2 \otimes \mathcal{Y}_3 + \mathrm{Ker}\ \mathbf{PP})))\\
&\qquad\qquad (w_1, w_2, w_3, w_4; y_1, y_2, x_3)\rangle_W\big|_{y_1^n = e^{n\log(z_1-z_3)}, y_2^n = e^{n\log(z_2-z_3)}, x_3^n = e^{n\log z_3}}\\
&\quad = \langle w', (\widetilde{\mathbf{PI}}(\mathcal{F}_{23}^{(1)}(\mathcal{Y}_1 \otimes \mathcal{Y}_2 \otimes \mathcal{Y}_3 + \mathrm{Ker}\ \mathbf{PP})))\\
&\qquad\qquad (w_1, w_2, w_3, w_4; x_1, y_2, x_3)\rangle_W\big|_{x_1^n = e^{n\log z_1}, y_2^n = e^{n\log(z_2-z_3)}, x_3^n = e^{n\log z_3}}\\
&\quad = \langle w', (\widetilde{\mathbf{PP}}(\mathcal{Y}_1 \otimes \mathcal{Y}_2 \otimes \mathcal{Y}_3 + \mathrm{Ker}\ \mathbf{PP}))\\
&\qquad\qquad (w_1, w_2, w_3, w_4; x_1, x_2, x_3)\rangle_W\big|_{x_1^n = e^{n\log z_1}, x_2^n = e^{n\log z_2}, x_3^n = e^{n\log z_3}}
\end{aligned}
\tag{4.73}
$$

for $w_1, w_2, w_3, w_4 \in W$ and $w' \in W'$ on the region $R = \{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid \mathrm{Re} z_1 > \mathrm{Re} z_2 > \mathrm{Re} z_3 > \mathrm{Re}(z_1 - z_3) > \mathrm{Re}(z_2 - z_3) > \mathrm{Re}(z_1 - z_2) > 0,\ \mathrm{Im} z_1 > \mathrm{Im} z_2 > \mathrm{Im} z_3 > \mathrm{Im}(z_1 - z_3) > \mathrm{Im}(z_2 - z_3) > \mathrm{Im}(z_1 - z_2) > 0\}$. So we have

$$
\begin{aligned}
&\langle w', (\widetilde{\mathbf{II}}((\mathcal{F}_{23}^{(2)} \circ \mathcal{F}_{12}^{(1)} - \mathcal{F}_{12}^{(2)} \circ \mathcal{F}_{13} \circ \mathcal{F}_{23}^{(1)})(\mathcal{Y}_1 \otimes \mathcal{Y}_2 \otimes \mathcal{Y}_3 + \mathrm{Ker}\ \mathbf{PP})))\\
&\qquad (w_1, w_2, w_3, w_4; x_0, y_2, x_3)\rangle_W\big|_{x_0^n = e^{n\log(z_1-z_2)}, y_2^n = e^{n\log(z_2-z_3)}, x_3^n = e^{n\log z_3}}\\
&\quad = 0
\end{aligned}
\tag{4.74}
$$

for $w_1, w_2, w_3, w_4 \in W$ and $w' \in W'$ on the region R. Note that the left-hand side of (4.74) is analytic in z_1, z_2 and z_3. Also note that R is a subset of the domain of this analytic function. From the basic properties of analytic functions, the left-hand side of (4.74) as an analytic function is 0 for all (z_1, z_2, z_3) in its domain. Thus for $w_1, w_2, w_3, w_4 \in W$ and $w' \in W'$,

$$
\begin{aligned}
&\langle w', (\widetilde{\mathbf{II}}((\mathcal{F}_{23}^{(2)} \circ \mathcal{F}_{12}^{(1)} - \mathcal{F}_{12}^{(2)} \circ \mathcal{F}_{13} \circ \mathcal{F}_{23}^{(1)})(\mathcal{Y}_1 \otimes \mathcal{Y}_2 \otimes \mathcal{Y}_3 + \mathrm{Ker}\ \mathbf{PP})))\\
&\qquad\qquad (w_1, w_2, w_3, w_4; x_0, y_2, x_3)\rangle_W = 0.
\end{aligned}
\tag{4.75}
$$

By the definition of the map $\widetilde{\mathbf{II}}$, (4.75) can be written as

$$
\widetilde{\mathbf{II}}((\mathcal{F}_{23}^{(2)} \circ \mathcal{F}_{12}^{(1)} - \mathcal{F}_{12}^{(2)} \circ \mathcal{F}_{13} \circ \mathcal{F}_{23}^{(1)})(\mathcal{Y}_1 \otimes \mathcal{Y}_2 \otimes \mathcal{Y}_3 + \mathrm{Ker}\ \mathbf{PP})) = 0. \tag{4.76}
$$

Thus

$$
\mathcal{F}_{23}^{(2)} \circ \mathcal{F}_{12}^{(1)}(\mathcal{Y}_1 \otimes \mathcal{Y}_2 \otimes \mathcal{Y}_3 + \mathrm{Ker}\ \mathbf{PP}) = \mathcal{F}_{12}^{(2)} \circ \mathcal{F}_{13} \circ \mathcal{F}_{23}^{(1)}(\mathcal{Y}_1 \otimes \mathcal{Y}_2 \otimes \mathcal{Y}_3 + \mathrm{Ker}\ \mathbf{PP}). \tag{4.77}
$$

So (4.71) holds. By linearity, we see that

$$
\mathcal{F}_{23}^{(2)} \circ \mathcal{F}_{12}^{(1)} = \mathcal{F}_{12}^{(2)} \circ \mathcal{F}_{13} \circ \mathcal{F}_{23}^{(1)}, \tag{4.78}
$$

which proves eq. (4.66).

Next we shall prove eq. (4.67).

Note that $\mathcal{F}\circ\tilde{\Omega}^{(3)}\circ\mathcal{F}$ and $\tilde{\Omega}^{(1)}\circ\mathcal{F}\circ\tilde{\Omega}^{(4)}$ are both isomorphisms from

$$\frac{\coprod_{a_1,a_2,a_3,a_4,a_5\in\mathcal{A}} \mathcal{V}_{a_1a_5}^{a_4}\otimes\mathcal{V}_{a_2a_3}^{a_5}}{\mathrm{Ker}\,\mathbf{P}} \tag{4.79}$$

to

$$\frac{\coprod_{a_1,a_2,a_3,a_4,a_5\in\mathcal{A}} \mathcal{V}_{a_3a_1}^{a_5}\otimes\mathcal{V}_{a_5a_2}^{a_4}}{\mathrm{Ker}\,\mathbf{I}}. \tag{4.80}$$

By linearity, it suffices to prove that

$$\mathcal{F}\circ\tilde{\Omega}^{(3)}\circ\mathcal{F}(\mathcal{Y}_1\otimes\mathcal{Y}_2+\mathrm{Ker}\,\mathbf{P})=\tilde{\Omega}^{(1)}\circ\mathcal{F}\circ\tilde{\Omega}^{(4)}(\mathcal{Y}_1\otimes\mathcal{Y}_2+\mathrm{Ker}\,\mathbf{P}) \tag{4.81}$$

for any $a_1,\cdots,a_5\in\mathcal{A}$, $\mathcal{Y}_1\in\mathcal{V}_{a_1a_5}^{a_4}$ and $\mathcal{Y}_2\in\mathcal{V}_{a_2a_3}^{a_5}$. Fix any $a_1,\cdots,a_5\in\mathcal{A}$, $\mathcal{Y}_1\in\mathcal{V}_{a_1a_5}^{a_4}$ and $\mathcal{Y}_2\in\mathcal{V}_{a_2a_3}^{a_5}$. Consider the simply connected region

$$\begin{aligned}\mathfrak{G} \;=\;& \mathbb{C}^2\backslash\big(\{(z_1,z_2)\in\mathbb{C}^2\mid z_1\in(-\infty,0]\}\cup\{(z_1,z_2)\in\mathbb{C}^2\mid z_2\in(-\infty,0]\}\\ &\cup\{(z_1,z_2)\in\mathbb{C}^2\mid z_1-z_2\in[0,+\infty)\}\big).\end{aligned}$$

Let $w_1,w_2,w_3\in W$ and $w'\in W'$ be any fixed elements. Then

$$\psi(z_1,z_2)=\langle w',(\tilde{\mathbf{P}}(\mathcal{Y}_1\otimes\mathcal{Y}_2+\mathrm{Ker}\,\mathbf{P}))(w_1,w_2,w_3;x_1,x_2)\rangle_W\big|_{x_1^n=e^{n\log z_1},x_2^n=e^{n\log z_2}} \tag{4.82}$$

on the region $S_1=\{(z_1,z_2)\in\mathbb{C}^2\mid \mathrm{Re}z_1>\mathrm{Re}z_2>\mathrm{Re}(z_1-z_2)>0,\ \mathrm{Im}z_1>\mathrm{Im}z_2>\mathrm{Im}(z_1-z_2)>0\}\subset\mathfrak{G}$ is a single-valued analytic function. By the convergence properties of intertwining operator algebras, (ψ,S_1) can be extended to a single-valued analytic function on $\mathfrak{G}$. In the following, we shall prove (4.81) by the analytic continuations of (ψ,S_1) along curves.

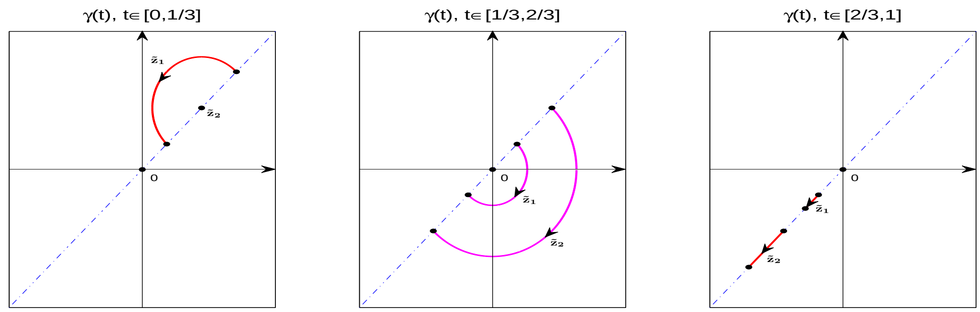
Let (a_0,b_0), (a_1,b_1) and (a_2,b_2) be three pairs of fixed positive real numbers satisfying

$$a_0>b_0>a_0-b_0>0,\quad a_1>a_1-b_1>b_1>0,\quad b_2>b_2-a_2>a_2>0. \tag{4.83}$$

Define a path $\gamma:\ [0,1]\to\mathfrak{G}$ by

$$\begin{aligned}\gamma(t)&=(\tilde{z}_1(t),\tilde{z}_2(t))\\ &=\begin{cases}\Big(b_0e^{\frac14\pi i}+(a_0-b_0)e^{\frac14\pi i+3t\pi i},\ \ b_0e^{\frac14\pi i}\Big) & t\in[0,\frac13],\\ \Big((2b_0-a_0)e^{\frac14\pi i-(3t-1)\pi i},\ \ b_0e^{\frac14\pi i-(3t-1)\pi i}\Big) & t\in(\frac13,\frac23],\\ \Big((3(2b_0-a_0)(1-t)+(3t-2)a_2)e^{-\frac34\pi i}, & \\ \qquad (3b_0(1-t)+(3t-2)b_2)e^{-\frac34\pi i}\Big) & t\in(\frac23,1].\end{cases}\end{aligned} \tag{4.84}$$

Figure 1: $\gamma(t)$



See Figure 1 for an illustration. Then $\gamma(t) \subset \mathfrak{G} \cap \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_2| > |z_1 - z_2| > 0\}$. For each $t \in [0, 1]$, we choose a simply connected region

$$D_t = \{(z_1, z_2) \in \mathbb{C}^2 \mid \max(|z_1 - \tilde{z}_1(t)|, |z_2 - \tilde{z}_2(t)|) < \varepsilon_t\}, \tag{4.85}$$

where ε_t is a sufficiently small positive real number for each $t \in [0, 1]$ such that

$$D_0 \subset \mathfrak{G} \cap \{(z_1, z_2) \in \mathbb{C}^2 \mid \mathrm{Re} z_1 > \mathrm{Re} z_2 > \mathrm{Re}(z_1 - z_2) > 0,\ \mathrm{Im} z_1 > \mathrm{Im} z_2 > \mathrm{Im}(z_1 - z_2) > 0\},$$

$$D_t \subset \mathfrak{G} \cap \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_2| > |z_1 - z_2| > 0\} \quad \text{for } t \in (0, 1),$$

$$D_1 \subset \mathfrak{G} \cap \{(z_1, z_2) \in \mathbb{C}^2 \mid \mathrm{Re} z_2 < \mathrm{Re}(z_2 - z_1) < \mathrm{Re} z_1 < 0,\ \mathrm{Im} z_2 < \mathrm{Im}(z_2 - z_1) < \mathrm{Im} z_1 < 0\}.$$

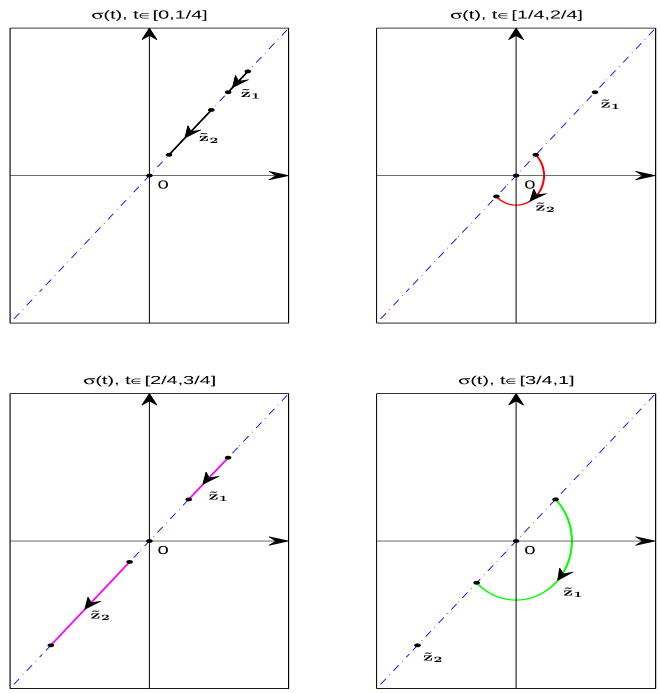
The existence of ε_t can be easily verified with some straightforward calculations, which shall be omitted here. Then

$$f_t = \langle e^{-x_2 L(1)} w', (\tilde{\mathbf{P}}(\tilde{\Omega}^{(3)} \mathcal{F}(\mathcal{Y}_1 \otimes \mathcal{Y}_2 + \mathrm{Ker}\,\mathbf{P})))(w_3, w_1, w_2; x_2, x_0)\rangle_W \Big|_{\substack{x_0^n = e^{n \log(z_1 - z_2)} \\ x_2^n = e^{n \log(-z_2)}}} \tag{4.86}$$

is a single-valued analytic function on the region D_t for each $t \in [0, 1]$. So we obtain an analytic continuation along γ: $\{(f_t, D_t) : 0 \le t \le 1\}$. Moreover, it can be derived from the fusing isomorphism and the skew-symmetry property that on the region D_0,

$$\begin{aligned}
f_0 &= \langle e^{-x_2 L(1)} w', (\tilde{\mathbf{P}}(\tilde{\Omega}^{(3)} \mathcal{F}(\mathcal{Y}_1 \otimes \mathcal{Y}_2 + \mathrm{Ker}\,\mathbf{P})))(w_3, w_1, w_2; x_2, x_0)\rangle_W \Big|_{\substack{x_0^n = e^{n \log(z_1 - z_2)} \\ x_2^n = e^{n \log(-z_2)}}} \\
&= \langle w', e^{x_2 L(-1)} (\tilde{\mathbf{P}}(\tilde{\Omega}^{(3)} \mathcal{F}(\mathcal{Y}_1 \otimes \mathcal{Y}_2 + \mathrm{Ker}\,\mathbf{P})))(w_3, w_1, w_2; e^{\pi i} x_2, x_0)\rangle_W \Big|_{\substack{x_0^n = e^{n \log(z_1 - z_2)} \\ x_2^n = e^{n \log z_2}}} \\
&= \langle w', (\tilde{\mathbf{I}}((\widetilde{\Omega^{-1}})^{(2)} \tilde{\Omega}^{(3)} \mathcal{F}(\mathcal{Y}_1 \otimes \mathcal{Y}_2 + \mathrm{Ker}\,\mathbf{P})))(w_1, w_2, w_3; x_0, x_2)\rangle_W \Big|_{\substack{x_0^n = e^{n \log(z_1 - z_2)} \\ x_2^n = e^{n \log z_2}}} \\
&= \langle w', (\tilde{\mathbf{I}}(\mathcal{F}(\mathcal{Y}_1 \otimes \mathcal{Y}_2 + \mathrm{Ker}\,\mathbf{P})))(w_1, w_2, w_3; x_0, x_2)\rangle_W \Big|_{\substack{x_0^n = e^{n \log(z_1 - z_2)} \\ x_2^n = e^{n \log z_2}}} \\
&= \langle w', (\tilde{\mathbf{P}}(\mathcal{Y}_1 \otimes \mathcal{Y}_2 + \mathrm{Ker}\,\mathbf{P}))(w_1, w_2, w_3; x_1, x_2)\rangle_W \Big|_{x_1^n = e^{n \log z_1}, x_2^n = e^{n \log z_2}},
\end{aligned} \tag{4.87}$$

Figure 2: $\sigma(t)$



and that on the region D_1,

$$\begin{aligned} f_1 &= \langle e^{-x_2 L(1)} w', (\tilde{\mathbf{P}}(\tilde{\Omega}^{(3)} \mathcal{F}(\mathcal{Y}_1 \otimes \mathcal{Y}_2 + \mathrm{Ker}\, \mathbf{P})))(w_3, w_1, w_2; x_2, x_0)\rangle_W \Big|_{\substack{x_0^n = e^{n \log(z_1 - z_2)} \\ x_2^n = e^{n\log(-z_2)}}} \\ &= \langle e^{-x_2 L(1)} w', (\tilde{\mathbf{I}}(\mathcal{F}\tilde{\Omega}^{(3)} \mathcal{F}(\mathcal{Y}_1 \otimes \mathcal{Y}_2 + \mathrm{Ker}\, \mathbf{P})))(w_3, w_1, w_2; x_1, x_0)\rangle_W \Big|_{\substack{x_0^n = e^{n \log(z_1 - z_2)} \\ x_1^n = e^{n\log(-z_1)} \\ x_2^n = e^{n\log(-z_2)}}} . \end{aligned} \tag{4.88}$$

Define another path $\sigma: \ [0,1] \to \mathfrak{G}$ by

$$\begin{aligned} \sigma(t) &= (\tilde{z}_1(t), \tilde{z}_2(t)) \\ &= \begin{cases} \Big((a_0(1-4t) + 4a_1 t)e^{\frac{1}{4}\pi i}, \ \ (b_0(1-4t)+4b_1 t)e^{\frac{1}{4}\pi i}\Big) & t \in [0, \frac{1}{4}], \\ \Big(a_1 e^{\frac{1}{4}\pi i}, \ \ b_1 e^{\frac{1}{4}\pi i - (4t-1)\pi i}\Big) & t \in (\frac{1}{4}, \frac{2}{4}], \\ \Big((a_1(3-4t) + a_2(4t-2))e^{\frac{1}{4}\pi i}, \ \ (b_1(3-4t) + b_2(4t-2))e^{-\frac{3}{4}\pi i}\Big) & t \in (\frac{2}{4}, \frac{3}{4}], \\ \Big(a_2 e^{\frac{1}{4}\pi i - (4t-3)\pi i}, \ \ b_2 e^{-\frac{3}{4}\pi i}\Big) & t \in (\frac{3}{4}, 1]. \end{cases} \end{aligned} \tag{4.89}$$

See Figure 2 for an illustration. Then $\sigma(t) \subset \mathfrak{G}$ and $\sigma(0) = \gamma(0)$, $\sigma(1) = \gamma(1)$. For each $t \in [0,1]$, we choose a simply connected region

$$E_t = \{(z_1, z_2) \in \mathbb{C}^2 \mid \max(|z_1 - \tilde{z}_1(t)|, |z_2 - \tilde{z}_2(t)|) < \epsilon_t\}, \tag{4.90}$$

where $E_0 = D_0$, $E_1 = D_1$ (i.e. $\epsilon_0 = \varepsilon_0$, $\epsilon_1 = \varepsilon_1$), and ϵ_t is a sufficiently small positive real number for each $t \in (0,1)$ such that

$$E_t \subset \mathfrak{G} \cap \{(z_1, z_2) \in \mathbb{C}^2 \mid \mathrm{Re} z_1 > \mathrm{Re} z_2 > 0,\ \mathrm{Im} z_1 > \mathrm{Im} z_2 > 0\} \quad \text{for } t \in (0, \frac{1}{4}),$$

$$E_{\frac{1}{4}} \subset \mathfrak{G} \cap \{(z_1, z_2) \in \mathbb{C}^2 \mid \mathrm{Re} z_1 > \mathrm{Re}(z_1 - z_2) > \mathrm{Re} z_2 > 0,\ \mathrm{Im} z_1 > \mathrm{Im}(z_1 - z_2) > \mathrm{Im} z_2 > 0\},$$

$$E_t \subset \mathfrak{G} \cap \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1 - z_2| > |z_2| > 0\} \quad \text{for } t \in (\frac{1}{4}, \frac{3}{4}),$$

$$E_{\frac{3}{4}} \subset \mathfrak{G} \cap \{(z_1, z_2) \in \mathbb{C}^2 \mid \mathrm{Re}(z_1 - z_2) > -\mathrm{Re} z_2 > \mathrm{Re} z_1 > 0,\ \mathrm{Im}(z_1 - z_2) > -\mathrm{Im} z_2 > \mathrm{Im} z_1 > 0\},$$

$$E_t \subset \mathfrak{G} \cap \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1 - z_2| > |z_1| > 0\} \quad \text{for } t \in (\frac{3}{4}, 1).$$

Then

$$g_t = \langle w', (\tilde{\mathbf{P}}(\mathcal{Y}_1 \otimes \mathcal{Y}_2 + \mathrm{Ker}\, \mathbf{P}))(w_1, w_2, w_3; x_1, x_2)\rangle_W \Big|_{\substack{x_1^n = e^{n \log z_1} \\ x_2^n = e^{n \log z_2}}} \tag{4.91}$$

is a single-valued analytic function on the region E_t for each $t \in [0, \frac{1}{4}]$;

$$g_t = \langle e^{-x_2 L(1)} w', (\tilde{\mathbf{P}}(\tilde{\Omega}^{(4)}(\mathcal{Y}_1 \otimes \mathcal{Y}_2 + \mathrm{Ker}\, \mathbf{P})))(w_1, w_3, w_2; x_0, x_2)\rangle_W \Big|_{\substack{x_0^n = e^{n \log(z_1 - z_2)} \\ x_2^n = e^{n \log(-z_2)}}} \tag{4.92}$$

is a single-valued analytic function on the region E_t for each $t \in (\frac{1}{4}, \frac{3}{4}]$; and

$$\begin{aligned} g_t = \langle e^{-x_2 L(1)} w', (\tilde{\mathbf{I}}(\tilde{\Omega}^{(1)} \mathcal{F} \tilde{\Omega}^{(4)}(\mathcal{Y}_1 \otimes \mathcal{Y}_2 + \mathrm{Ker}\, \mathbf{P}))) \\ (w_3, w_1, w_2; x_1, x_0)\rangle_W \big|_{x_1^n = e^{n \log(-z_1)}, x_0^n = e^{n \log(z_1 - z_2)}, x_2^n = e^{n \log(-z_2)}} \end{aligned} \tag{4.93}$$

is a single-valued analytic function on the region E_t for each $t \in (\frac{3}{4}, 1]$. Moreover, it can be derived from the fusing isomorphism and the skew-symmetry property that on the region $E_{\frac{1}{4}}$,

$$\begin{aligned} g_{\frac{1}{4}} &= \langle w', (\tilde{\mathbf{P}}(\mathcal{Y}_1 \otimes \mathcal{Y}_2 + \mathrm{Ker}\, \mathbf{P}))(w_1, w_2, w_3; x_1, x_2)\rangle_W \Big|_{\substack{x_1^n = e^{n \log z_1} \\ x_2^n = e^{n \log z_2}}} \\ &= \langle w', (\tilde{\mathbf{P}}((\widetilde{\Omega^{-1}})^{(4)} \tilde{\Omega}^{(4)}(\mathcal{Y}_1 \otimes \mathcal{Y}_2 + \mathrm{Ker}\, \mathbf{P})))(w_1, w_2, w_3; x_1, x_2)\rangle_W \Big|_{\substack{x_1^n = e^{n \log z_1} \\ x_2^n = e^{n \log z_2}}} \\ &= \langle w', e^{x_2 L(-1)} (\tilde{\mathbf{P}}(\tilde{\Omega}^{(4)}(\mathcal{Y}_1 \otimes \mathcal{Y}_2 + \mathrm{Ker}\, \mathbf{P})))(w_1, w_3, w_2; x_0, e^{\pi i} x_2)\rangle_W \Big|_{\substack{x_0^n = e^{n \log(z_1 - z_2)} \\ x_2^n = e^{n \log z_2}}} \\ &= \langle e^{-x_2 L(1)} w', (\tilde{\mathbf{P}}(\tilde{\Omega}^{(4)}(\mathcal{Y}_1 \otimes \mathcal{Y}_2 + \mathrm{Ker}\, \mathbf{P})))(w_1, w_3, w_2; x_0, x_2)\rangle_W \Big|_{\substack{x_0^n = e^{n \log(z_1 - z_2)} \\ x_2^n = e^{n \log(-z_2)}}}, \end{aligned} \tag{4.94}$$

and that on the region $E_{\frac{3}{4}}$,

$$\begin{aligned}
g_{\frac{3}{4}} &= \langle e^{-x_2L(1)}w', (\tilde{\mathbf{P}}(\tilde{\Omega}^{(4)}(\mathcal{Y}_1\otimes\mathcal{Y}_2+\mathrm{Ker}\,\mathbf{P})))(w_1,w_3,w_2;x_0,x_2)\rangle_W\Big|_{\substack{x_0^n=e^{n\log(z_1-z_2)}\\ x_2^n=e^{n\log(-z_2)}}}\\
&= \langle e^{-x_2L(1)}w', (\tilde{\mathbf{I}}(\mathcal{F}\tilde{\Omega}^{(4)}(\mathcal{Y}_1\otimes\mathcal{Y}_2+\mathrm{Ker}\,\mathbf{P})))(w_1,w_3,w_2;x_1,x_2)\rangle_W\Big|_{\substack{x_1^n=e^{n\log z_1}\\ x_2^n=e^{n\log(-z_2)}}}\\
&= \langle e^{-x_2L(1)}w', (\tilde{\mathbf{I}}((\widetilde{\Omega^{-1}})^{(1)}\tilde{\Omega}^{(1)}\mathcal{F}\tilde{\Omega}^{(4)}(\mathcal{Y}_1\otimes\mathcal{Y}_2+\mathrm{Ker}\,\mathbf{P})))\\
&\qquad (w_1,w_3,w_2;x_1,x_2)\rangle_W\big|_{x_1^n=e^{n\log z_1},x_2^n=e^{n\log(-z_2)}}\\
&= \langle e^{-x_2L(1)}w', (\tilde{\mathbf{I}}(\tilde{\Omega}^{(1)}\mathcal{F}\tilde{\Omega}^{(4)}(\mathcal{Y}_1\otimes\mathcal{Y}_2+\mathrm{Ker}\,\mathbf{P})))\\
&\qquad (w_3,w_1,w_2;e^{\pi i}x_1,x_0)\rangle_W\big|_{x_1^n=e^{n\log z_1},x_0^n=e^{n\log(z_1-z_2)},x_2^n=e^{n\log(-z_2)}}\\
&= \langle e^{-x_2L(1)}w', (\tilde{\mathbf{I}}(\tilde{\Omega}^{(1)}\mathcal{F}\tilde{\Omega}^{(4)}(\mathcal{Y}_1\otimes\mathcal{Y}_2+\mathrm{Ker}\,\mathbf{P})))\\
&\qquad (w_3,w_1,w_2;x_1,x_0)\rangle_W\big|_{x_1^n=e^{n\log(-z_1)},x_0^n=e^{n\log(z_1-z_2)},x_2^n=e^{n\log(-z_2)}},
\end{aligned}\tag{4.95}$$

where the second equation of (4.95) is obtained by changing the variables (z_1,z_2) by $(z_1-z_2,-z_2)$ in (2.110). So $\{(g_t,E_t):0\le t\le 1\}$ is an analytic continuation along σ.

Since $\mathfrak{G}$ is simply connected, and $\sigma,\gamma\subset\mathfrak{G}$, $\sigma(0)=\gamma(0)$, $\sigma(1)=\gamma(1)$, we can derive that the two paths σ,γ are homotopic. Moreover, from (4.87) we see that $(f_0,D_0)=(g_0,D_0)$. So $f_1=g_1$ on the region $D_1\cap E_1=D_1=E_1$. Namely, with (4.88) we see that

$$\begin{aligned}
&\langle e^{-x_2L(1)}w', (\tilde{\mathbf{I}}(\mathcal{F}\tilde{\Omega}^{(3)}\mathcal{F}(\mathcal{Y}_1\otimes\mathcal{Y}_2+\mathrm{Ker}\,\mathbf{P})))\\
&\qquad\qquad (w_3,w_1,w_2;x_1,x_0)\rangle_W\big|_{x_1^n=e^{n\log(-z_1)},x_0^n=e^{n\log(z_1-z_2)},x_2^n=e^{n\log(-z_2)}}\\
&\quad= \langle e^{-x_2L(1)}w', (\tilde{\mathbf{I}}(\tilde{\Omega}^{(1)}\mathcal{F}\tilde{\Omega}^{(4)}(\mathcal{Y}_1\otimes\mathcal{Y}_2+\mathrm{Ker}\,\mathbf{P})))\\
&\qquad\qquad (w_3,w_1,w_2;x_1,x_0)\rangle_W\big|_{x_1^n=e^{n\log(-z_1)},x_0^n=e^{n\log(z_1-z_2)},x_2^n=e^{n\log(-z_2)}}
\end{aligned}\tag{4.96}$$

on the region $D_1=E_1$. Since both hand sides of (4.96) are analytic functions on the domain $\{(z_1,z_2)\in\mathbb{C}^2\mid |z_1-z_2|>|z_1|>0\}$ which contains $D_1=E_1$, we have

$$\begin{aligned}
&\langle e^{-x_2L(1)}w', (\tilde{\mathbf{I}}(\mathcal{F}\tilde{\Omega}^{(3)}\mathcal{F}(\mathcal{Y}_1\otimes\mathcal{Y}_2+\mathrm{Ker}\,\mathbf{P})))\\
&\qquad\qquad (w_3,w_1,w_2;x_1,x_0)\rangle_W\big|_{x_1^n=e^{n\log(-z_1)},x_0^n=e^{n\log(z_1-z_2)},x_2^n=e^{n\log(-z_2)}}\\
&\quad= \langle e^{-x_2L(1)}w', (\tilde{\mathbf{I}}(\tilde{\Omega}^{(1)}\mathcal{F}\tilde{\Omega}^{(4)}(\mathcal{Y}_1\otimes\mathcal{Y}_2+\mathrm{Ker}\,\mathbf{P})))\\
&\qquad\qquad (w_3,w_1,w_2;x_1,x_0)\rangle_W\big|_{x_1^n=e^{n\log(-z_1)},x_0^n=e^{n\log(z_1-z_2)},x_2^n=e^{n\log(-z_2)}}
\end{aligned}\tag{4.97}$$

on the domain $\{(z_1,z_2)\in\mathbb{C}^2\mid |z_1-z_2|>|z_1|>0\}$. Hence,

$$\begin{aligned}
&\langle e^{-x_2L(1)}w', (\tilde{\mathbf{I}}(\mathcal{F}\tilde{\Omega}^{(3)}\mathcal{F}(\mathcal{Y}_1\otimes\mathcal{Y}_2+\mathrm{Ker}\,\mathbf{P})))(w_3,w_1,w_2;x_1,x_0)\rangle_W\\
&\qquad -\langle e^{-x_2L(1)}w', (\tilde{\mathbf{I}}(\tilde{\Omega}^{(1)}\mathcal{F}\tilde{\Omega}^{(4)}(\mathcal{Y}_1\otimes\mathcal{Y}_2+\mathrm{Ker}\,\mathbf{P})))(w_3,w_1,w_2;x_1,x_0)\rangle_W\\
&\quad=0
\end{aligned}\tag{4.98}$$

for any $w_1, w_2, w_3 \in W$ and $w' \in W'$, which further implies

$$\mathcal{F}\tilde{\Omega}^{(3)}\mathcal{F}(\mathcal{Y}_1 \otimes \mathcal{Y}_2 + \mathrm{Ker}\ \mathbf{P}) = \tilde{\Omega}^{(1)}\mathcal{F}\tilde{\Omega}^{(4)}(\mathcal{Y}_1 \otimes \mathcal{Y}_2 + \mathrm{Ker}\ \mathbf{P}). \tag{4.99}$$

So eq. (4.81) holds, therefore proving (4.67).

Eq. (4.68) can be proved similarly; we omit the details here. ∎

Moreover, by (4.64), (4.65), (4.67) and (4.68), we have another two relations:

Corollary 4.3. *The fusing isomorphism and the skew-symmetry isomorphisms satisfy the following genus-zero Moore-Seiberg equations:*

$$\mathcal{F}^{-1} \circ \tilde{\Omega}^{(2)} \circ \mathcal{F}^{-1} = \tilde{\Omega}^{(4)} \circ \mathcal{F}^{-1} \circ \tilde{\Omega}^{(1)}, \tag{4.100}$$

$$\mathcal{F}^{-1} \circ (\widetilde{\Omega^{-1}})^{(2)} \circ \mathcal{F}^{-1} = (\widetilde{\Omega^{-1}})^{(4)} \circ \mathcal{F}^{-1} \circ (\widetilde{\Omega^{-1}})^{(1)}. \tag{4.101}$$

We call (4.66) the *pentagon identity* and (4.67), (4.68), (4.100) and (4.101) the *hexagon identities* since they correspond to the commutativity of the pentagon and hexagon diagrams for braided tensor categories.

5 Intertwining operator algebras in terms of the Jacobi identity

In this section, we first give another definition of intertwining operator algebras in terms of the Jacobi identity (cf. [H9]), then we prove the equivalence of this definition and the definition in Section 2.

Let $\mathbf{P}$ and $\mathbf{I}$ be the multiplication and iterates of intertwining operators, respectively (cf. (2.59), (2.64)). Then the definition of intertwining operator algebras in terms of the Jacobi identity is as follows:

Definition 5.1 (**Intertwining operator algebra).** An *intertwining operator algebra of central charge* $c \in \mathbb{C}$ is a vector space

$$W = \coprod_{a \in \mathcal{A}} W^a \tag{5.1}$$

graded by a finite set $\mathcal{A}$ containing a special element e (graded by *color*), equipped with a vertex operator algebra structure of central charge c on W^e, a W^e-module structure on W^a for each $a \in \mathcal{A}$, a subspace $\mathcal{V}_{a_1 a_2}^{a_3}$ of the space of all intertwining operators of type $\binom{W^{a_3}}{W^{a_1} W^{a_2}}$ for each triple $a_1, a_2, a_3 \in \mathcal{A}$, an isomorphism

$$\mathcal{F} : \quad \frac{\coprod_{a_1,a_2,a_3,a_4,a_5 \in \mathcal{A}} \mathcal{V}_{a_1 a_5}^{a_4} \otimes \mathcal{V}_{a_2 a_3}^{a_5}}{\mathrm{Ker}\ \mathbf{P}} \longrightarrow \frac{\coprod_{a_1,a_2,a_3,a_4,a_5 \in \mathcal{A}} \mathcal{V}_{a_1 a_2}^{a_5} \otimes \mathcal{V}_{a_5 a_3}^{a_4}}{\mathrm{Ker}\ \mathbf{I}} \tag{5.2}$$

satisfying

$$\mathcal{F}(\pi_P(\coprod_{a_5\in\mathcal{A}} \mathcal{V}_{a_1a_5}^{a_4}\otimes\mathcal{V}_{a_2a_3}^{a_5})) = \pi_I(\coprod_{a_5\in\mathcal{A}} \mathcal{V}_{a_1a_2}^{a_5}\otimes\mathcal{V}_{a_5a_3}^{a_4}) \tag{5.3}$$

for each quadruple $a_1, a_2, a_3, a_4 \in \mathcal{A}$, and an isomorphism

$$\Omega(a_1, a_2; a_3) : \mathcal{V}_{a_1a_2}^{a_3} \to \mathcal{V}_{a_2a_1}^{a_3} \tag{5.4}$$

for each triple $a_1, a_2, a_3 \in \mathcal{A}$, satisfying the following axioms:

1. For any $a \in \mathcal{A}$, there exists $h_a \in \mathbb{R}$ such that the $\mathbb{C}$-graded module W^a is $h_a+\mathbb{Z}$-graded.

2. The W^e-module structure on W^e is the adjoint module structure. For any $a \in \mathcal{A}$, the space $\mathcal{V}_{ea}^a$ is the one-dimensional vector space spanned by the vertex operators for the W^e-module W^a, and for any $\mathcal{Y} \in \mathcal{V}_{ea}^a$ and any $w_{(e)} \in W^e$, $w_{(a)} \in W^a$,

$$((\Omega(e, a; a))(\mathcal{Y}))(w_{(a)}, x)w_{(e)} = e^{xL(-1)}\mathcal{Y}(w_{(e)}, -x)w_{(a)}. \tag{5.5}$$

   For any $a_1, a_2 \in \mathcal{A}$ such that $a_1 \neq a_2$, $\mathcal{V}_{ea_1}^{a_2} = 0$.

3. For any $m \in \mathbb{Z}_+$, $a_i, b_j \in \mathcal{A}$, $w_{(a_i)} \in W^{a_i}$, $\mathcal{Y}_i \in \mathcal{V}_{a_i\ b_{i+1}}^{b_i}$, $i = 1, \ldots, m$, $j = 1, \ldots, m+1$, $w'_{(b_1)} \in (W^{b_1})'$ and $w_{(b_{m+1})} \in W^{b_{m+1}}$, the series

$$\langle w'_{(b_1)}, \mathcal{Y}_1(w_{(a_1)}, x_1)\cdots\mathcal{Y}_m(w_{(a_m)}, x_m)w_{(b_{m+1})}\rangle_{W^{b_1}}|_{x_i^n=e^{n\log z_i},\ i=1,\ldots,m,\ n\in\mathbb{R}} \tag{5.6}$$

   is absolutely convergent when $|z_1| > \cdots > |z_m| > 0$ and its sum can be analytically extended to a multivalued analytic function on the region given by $z_i \neq 0$, $i = 1, \ldots, m$, $z_i \neq z_j$, $i \neq j$, such that for any set of possible singular points with either $z_i = 0$, $z_i = \infty$ or $z_i = z_j$ for $i \neq j$, this multivalued analytic function can be expanded near the singularity as a series having the same form as the expansion near the singular points of a solution of a system of differential equations with regular singular points. For any $\mathcal{Y}_1 \in \mathcal{V}_{a_1a_2}^{a_5}$ and $\mathcal{Y}_2 \in \mathcal{V}_{a_5a_3}^{a_4}$, the series

$$\langle w'_{(a_4)}, \mathcal{Y}_2(\mathcal{Y}_1(w_{(a_1)}, x_0)w_{(a_2)}, x_2)w_{(a_3)}\rangle_{W^{a_4}}|_{x_0^n=e^{n\log(z_1-z_2)},\ x_2^n=e^{n\log z_2},\ n\in\mathbb{R}} \tag{5.7}$$

   is absolutely convergent when $|z_2| > |z_1 - z_2| > 0$.

4. Let

$$\Omega : \coprod_{a_1,a_2,a_3\in\mathcal{A}} \mathcal{V}_{a_1a_2}^{a_3} \to \coprod_{a_1,a_2,a_3\in\mathcal{A}} \mathcal{V}_{a_1a_2}^{a_3} \tag{5.8}$$

   be the isomorphism obtained from the isomorphism $\Omega(a_1, a_2; a_3)$ (cf. (5.4)) for all $a_1, a_2, a_3 \in \mathcal{A}$. For any $a_1, \cdots, a_5 \in \mathcal{A}$, $m \in \mathbb{N}$, $\mathcal{Y}_{a_1a_2,i}^{a_5} \in \mathcal{V}_{a_1a_2}^{a_5}$ and $\mathcal{Y}_{a_5a_3,i}^{a_4} \in \mathcal{V}_{a_5a_3}^{a_4}$,

$$\sum_{a_1,\cdots,a_5\in\mathcal{A}} \sum_{i=1}^{m} \mathcal{Y}_{a_1a_2,i}^{a_5} \otimes \mathcal{Y}_{a_5a_3,i}^{a_4} \in \mathrm{Ker}\ \mathbf{I} \tag{5.9}$$

if and only if

$$\sum_{a_1,\cdots,a_5\in\mathcal{A}}\sum_{i=1}^{m}\Omega(\mathcal{Y}^{a_5}_{a_1a_2,i})\otimes\mathcal{Y}^{a_4}_{a_5a_3,i}\in \text{Ker }\mathbf{I}. \tag{5.10}$$

5. The Jacobi identity: Let

$$\tilde{\Omega}^{(1)}:\ \frac{\coprod_{a_1,a_2,a_3,a_4,a_5\in\mathcal{A}}\mathcal{V}^{a_5}_{a_1a_2}\otimes\mathcal{V}^{a_4}_{a_5a_3}}{\text{Ker }\mathbf{I}}\longrightarrow\frac{\coprod_{a_1,a_2,a_3,a_4,a_5\in\mathcal{A}}\mathcal{V}^{a_5}_{a_2a_1}\otimes\mathcal{V}^{a_4}_{a_5a_3}}{\text{Ker }\mathbf{I}} \tag{5.11}$$

be the isomorphism determined by linearity and by

$$\tilde{\Omega}^{(1)}(\mathcal{Y}_1\otimes\mathcal{Y}_2+\text{Ker }\mathbf{I})=\Omega(\mathcal{Y}_1)\otimes\mathcal{Y}_2+\text{Ker }\mathbf{I} \tag{5.12}$$

for $\mathcal{Y}_1\in\mathcal{V}^{a_5}_{a_1a_2}$, $\mathcal{Y}_2\in\mathcal{V}^{a_4}_{a_5a_3}$ and $a_1,\ldots,a_5\in\mathcal{A}$. Let

$$\mathcal{B}=\mathcal{F}^{-1}\circ\tilde{\Omega}^{(1)}\circ\mathcal{F}:$$
$$\frac{\coprod_{a_1,a_2,a_3,a_4,a_5\in\mathcal{A}}\mathcal{V}^{a_4}_{a_1a_5}\otimes\mathcal{V}^{a_5}_{a_2a_3}}{\text{Ker }\mathbf{P}}\longrightarrow\frac{\coprod_{a_1,a_2,a_3,a_4,a_5\in\mathcal{A}}\mathcal{V}^{a_4}_{a_2a_5}\otimes\mathcal{V}^{a_5}_{a_1a_3}}{\text{Ker }\mathbf{P}}. \tag{5.13}$$

Then there exist linear maps $f^{a_1,a_2,a_3,a_4}_{\alpha}$, $g^{a_1,a_2,a_3,a_4}_{\alpha}$ and $h^{a_1,a_2,a_3,a_4}_{\alpha}$ of the form (2.127), (2.128) and (2.129), such that (2.131)-(2.136) and the Jacobi identity (2.137) hold for any $a_1,a_2,a_3,a_4\in\mathcal{A}$, $w_{(a_1)}\in W^{a_1}$, $w_{(a_2)}\in W^{a_2}$, $w_{(a_3)}\in W^{a_3}$, any

$$\mathcal{Z}\in\coprod_{a_5\in\mathcal{A}}\mathcal{V}^{a_4}_{a_1a_5}\otimes\mathcal{V}^{a_5}_{a_2a_3}\subset\coprod_{a_1,a_2,a_3,a_4,a_5\in\mathcal{A}}\mathcal{V}^{a_4}_{a_1a_5}\otimes\mathcal{V}^{a_5}_{a_2a_3}, \tag{5.14}$$

and any $\alpha\in\mathbb{A}$.

Theorem 5.2. *Definition 2.3 and Definition 5.1 are equivalent.*

Proof. Suppose $(W,\mathcal{A},\{\mathcal{V}^{a_3}_{a_1a_2}\},\mathbf{1},\omega)$ is a set of data satisfying the axioms of Definition 2.3. Then the results of the former sections imply that $(W,\mathcal{A},\{\mathcal{V}^{a_3}_{a_1a_2}\},\mathbf{1},\omega)$ satisfies the axioms of Definition 5.1.

Conversely, we suppose that $(W,\mathcal{A},\{\mathcal{V}^{a_3}_{a_1a_2}\},\mathbf{1},\omega)$ is a set of data satisfying the axioms of Definition 5.1. Then firstly, $(W,\mathcal{A},\{\mathcal{V}^{a_3}_{a_1a_2}\},\mathbf{1},\omega)$ satisfies the first three axioms of Definition 2.3. Secondly, the associativity in formal variables follows from the Jacobi identity (cf. Theorem 3.3), which immediately implies the associativity axiom of Definition 2.3 and (2.108)-(2.110). Thirdly, the commutativity property holds by the Jacobi identity (cf. Theorem 3.3). Then by the associativity, commutativity properties, (5.4), (5.5) and Theorem 3.2, we obtain the skew-symmetry of Definition 2.3. So in summary, $(W,\mathcal{A},\{\mathcal{V}^{a_3}_{a_1a_2}\},\mathbf{1},\omega)$ satisfies the axioms of Definition 2.3.

So the two definitions are equivalent. ∎

Remark 5.3. *In Definition 5.1, the isomorphism $\mathcal{F}$ in (5.2) is in fact the fusing isomorphism, the isomorphism Ω in (5.8) is in fact the skew-symmetry isomorphism, and the isomorphism $\mathcal{B}$ in (5.13) is in fact the braiding isomorphism.*

Proof. Since the Jacobi identity in Definition 5.1 implies associativity in formal variables and (2.108)-(2.110), by the definition of fusing isomorphism in Section 2, we can see that the isomorphism $\mathcal{F}$ in (5.2) coincides with the fusing isomorphism in (2.98). Moreover, by commutativity in formal variables (cf. Theorem 2.15) and the Jacobi identity in Definition 5.1, we see that the isomorphism $\mathcal{B}$ in (5.13) coincides with the braiding isomorphism in (2.106). Furthermore, since $\mathcal{F}$ in (5.2) and $\mathcal{B}$ in (5.13) are both isomorphisms, we obtain that $\tilde{\Omega}^{(1)}$ in (5.11) coincides with the isomorphism in (2.100). Then with $\tilde{\Omega}^{(1)}$ acting on $\mathcal{Y}_1 \otimes \mathcal{Y}_2 + \mathrm{Ker}\,\mathbf{I}$ for any $\mathcal{Y}_1 \in \mathcal{V}_{a_1 a_2}^{a_3}$, $0 \neq \mathcal{Y}_2 \in \mathcal{V}_{a_3 e}^{a_3}$, $a_1, a_2, a_3 \in \mathcal{A}$, we can derive that the isomorphism Ω in (5.8) coincides with the skew-symmetry isomorphism in (2.26). ∎

Moreover, one more definition of intertwining operator algebras was given in [H9], which added an explicit description of the vertex operator algebras. It is basically the same as Definition 5.1.

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